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POLINOMIOS DE HODGE DE VARIEDADES DE CARACTERES=
HODGE POLYNOMIALS OF CHARACTER VARIETIES

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Contents

Introducción

Esta tesis se dedica al estudio de una clase de invariantes algebraicos llamados E-polinomios, que pueden asociarse a cualquier variedad quasi-proyectiva X. Los E-polinomios (llamados también polinomios de Hodge-Deligne en la literatura) son de naturaleza cohomológica y contienen información topológica y aritmética de la variedad X. Nos centramos en un tipo particular de variedades algebraicas llamadas variedades de caracteres. Dado un grupo finitamente presentado Γ y un grupo algebraico reductivo G, las variedades de caracteres son el espacio de móduli de representaciones de Γ en G,

$$\mathcal{M}_\Gamma(G) := \mathrm{Hom}\,(\Gamma, G)//G$$

donde G actúa por conjugación. El cociente topológico del espacio de representaciones por la acción de G puede no tener buenas propiedades, por lo que se identifican algunas órbitas y, en este contexto algebraico, el espacio de móduli se construye utilizando Teoría de Invariantes Geométricos (GIT). Las variedades de caracteres son un objeto central en muchas ramas de la matemática: aparecen en dinámica, teoría de representaciones, geometría algebraica y diferencial... En esta introducción, nos centramos en los aspectos relevantes para situar el problema desarrollado en esta tesis en su contexto adecuado. El lector puede consultar el artículo [71] y las referencias que aparecen en él para un enfoque de las variedades de caracteres más global.

Estructuras geométricas

La obra de Felix Klein en el siglo XIX puso de manifiesto que las geometrías clásicas podían interpretarse como las propiedades de un espacio que son invariantes por una acción continua y transitiva (la acción de un grupo de Lie). Ejemplos como $S^n, \mathbb{C}P^n$... pueden ser vistos como espacios homogéneos: cocientes de su grupo de isometrías por un cierto subgrupo, el estabilizador de un punto. Una pregunta natural y algo menos rígida es su versión local. Dado un espacio topológico fijado Σ, ¿admite un sistema local de coordenadas a un espacio homogéneo fijado $X = G/H$ de modo que los cambios de coordenadas son isometrías (elementos de G)? ¿Con qué geometrías podemos dotar a un espacio cualquiera? Para responder a estas preguntas aparece la noción de (G, X)-estructuras en Σ.

Las variedades de caracteres surgen en este contexto como espacios de móduli que parametrizan clases de equivalencia de estas estructuras. Dado un atlas $\{(U_\alpha, \psi_\alpha)\}_{\alpha\in\Lambda}$, donde $\mathcal{U} = \{U_\alpha\}$ es un recubrimiento por abiertos de Σ y $\psi_\alpha : U_\alpha \to X$ son cartas locales, podemos considerar los cambios de coordenadas sobre conjuntos conexos, $g_{\alpha\beta}(U_\alpha, U_\beta) \in G$, que satisfacen que $\psi_\alpha = g_{\alpha\beta} \circ \psi_\beta$. El conjunto $\{g_{\alpha\beta}\}$ define un fibrado plano del siguiente modo: considérese el fibrado trivial E_α sobre cada U_α, $E_\alpha = U_\alpha \times X \to U_\alpha$, e identifíquese sobre las intersecciones $(u, x) \sim (u, g_{\alpha\beta}x)$. Este procedimiento define un fibrado $E \to \Sigma$ con fibra X y grupo de estructura G. El pullback del fibrado al recubridor universal $\tilde{\Sigma}$ nos da el fibrado trivial, y a partir de éste puede reconstruirse también el fibrado E. De modo más preciso, podemos recuperar E de $\tilde{\Sigma} \times X$ como un cociente por una acción de $\pi_1(\Sigma)$ que respete la acción en $\tilde{\Sigma}$ por transformaciones deck. Esta acción queda determinada por un homomorfismo $\pi_1(\Sigma) \to G$, la llamada *representación de la holonomía.* El hecho clave es que las clases de isomorfismo de fibrados planos se corresponden con elementos de $\mathrm{Hom}\,(\pi_1(\Sigma), G)$ modulo automorfismos internos de G. En otras palabras, existe una correspondencia entre los llamados sistemas G-locales (G-fibrados planos) y representaciones del grupo fundamental de Σ en G.

Por otro lado, los sistemas de coordenadas $\{\psi_\alpha : U_\alpha \to X\}$ pueden ser pegados para obtener una sección global del fibrado E, llamada la *sección de desarrollo* en la literatura (*developing section*). Puede ser interpretada como una aplicación $\pi_1(X)$-equivariante $\tilde{\Sigma} \xrightarrow{s} X$, donde $\pi_1(\Sigma)$ actúa en el último espacio mediante ρ. A medida que uno varía G, uno obtiene una geometría distinta, llamada *estructura localmente homogénea* en este contexto. Un ejemplo clásico es cuando $G = Isom(\mathbb{H}) \cong PGL(2, \mathbb{R})$ es el grupo de isometrías del plano hiperbólico; en este caso la correspondiente estructura geométrica se dice hiperbólica. Es aquí importante el espacio de Teichmuller, $\mathcal{T}(\Sigma)$, equivalente por el Teorema de Uniformización al conjunto de clases de equivalencia de estructuras hiperbólicas completas en Σ. Dada una estructura hiperbólica, la holonomía proporciona una representación de $\pi_1(\Sigma)$ en $PGL(2, \mathbb{R})$, bien definida salvo conjugación. Hay por tanto una aplicación:

$$hol : \mathcal{T}(\Sigma) \longrightarrow \mathrm{Hom}\,(\Sigma, PGL(2, \mathbb{R}))/PGL(2, \mathbb{R})$$

La aplicación es un embebimiento, y la imagen son aquellas representaciones que son discretas y fieles. Se corresponden con una componente conexa de la variedad de caracteres. En general, hay un espacio de deformaciones asociado a las (G, X)-estructuras y una aplicación de holonomía $hol : Def^{G,X}(\Sigma) \longrightarrow \mathrm{Hom}\,(\pi_1(\Sigma), G)//G$. Por ejemplo, cuando $G = PSL(2, \mathbb{C})$ se recuperan estructuras complejas proyectivas [23], o estructuras convexas proyectivas para $PSL(3, \mathbb{R})$ [12]. El estudio de invariantes topológicos que permitan distinguir y contar las componentes conexas de estos espacios es un tema que ha sido profusamente tratado [28, 9]. Por supuesto, muchas otras cuestiones que surgen aquí han sido estudiadas a lo largo de los años, como problemas dinámicos, relacionados con la acción del *mapping class group* ([29, 10]) o cuestiones de rigidez local de subgrupos de un grupo de Lie dado ([81, 70]).

El espacio de móduli de fibrados. Teoría de Hodge no abeliana

El caso en que $G = U(n)$ cobró relevancia gracias al trabajo de Narasimhan and Seshadri [63], nacido del deseo de entender los fibrados holomorfos sobre una superficie de Riemann. Los casos en que Σ es la esfera de Riemann o una curva elíptica habían sido ya estudiados por Grothendieck and Atiyah respectivamente [33],[2]. En su artículo de 1965, Narasimhan and Seshadri establecieron que el espacio de móduli de fibrados holomorfos estables de rango n y grado 0 sobre una superficie de Riemann compacta es difeomorfo al espacio de móduli de representaciones irreducibles de su grupo fundamental en $U(n)$, la $U(n)$-variedad de caracteres de Σ. La estructura topológica de un fibrado complejo E sobre Σ está determinada por su rango y su grado, y debido al teorema, su estructura diferenciable es independiente de la estructura compleja que tenga Σ. Si fijamos una estructura compleja en Σ, la parte $(0, 1)$ de una conexión unitaria A define una estructura holomorfa en E: una sección se dice holomofa si es anulada por la parte $(0, 1)$ de la derivada covariante asociada a A (todas son integrables, puesto que no existen formas de tipo $(0, 2)$ en Σ). Como la conexión es unitaria, su parte $(0, 1)$ determina toda A. Por último, la condición de estabilidad se corresponde con la irreducibilidad de la representación unitaria del grupo fundamental asociada, y conexiones equivalentes dan lugar a representaciones conjugadas. El espacio de móduli puede ser también descrito como un cociente simpléctico del espacio afín de conexiones del fibrado trivial, donde la aplicación momento viene dada por la curvatura [3]. En el caso abeliano, la variedad de caracteres es simplemente el toro complejo de dimensión g,

$$\mathrm{Hom}\,(\pi_1(\Sigma), U(1))/U(1) = \mathrm{Hom}\,(\pi_1(\Sigma), U(1)) \cong \mathrm{Hom}\,(H_1(\Sigma), U(1)) \cong U(1)^{2g+1}$$

que, si fijamos una estructura compleja en Σ, es precisamente el espacio de móduli de fibrados de línea holomorfos en Σ, la Jacobiana $\mathrm{Jac}\,(\Sigma) \cong \mathrm{Pic}^0(\Sigma)$.

Para fibrados E de grado d y rango n arbitrarios, existe una correspondencia similar, pero puesto que en ese caso E no admite conexiones planas, los fibrados holomorfos estables sobre Σ se corresponden a las llamadas conexiones proyectivamente planas, que dan lugar a representaciones de extensiones centrales $\tilde{\pi}_1(\Sigma)$ del grupo fundamental en $U(n)$. De modo explícito, la variedad de caracteres (twisted) correspondiente viene dada por

$$\operatorname{Hom}(\tilde{\pi}_1(\Sigma), U(n))/U(n) = \Big\{(A_1, B_1, \ldots, A_g, B_g, C) \in U(n)^{2g} \mid [A_i, C] = [B_i, C] = \mathrm{Id}, \prod_{i=1}^{g}[A_i, B_i] = C\Big\}$$

Si $\rho \in \operatorname{Hom}(\tilde{\pi}_1(\Sigma), U(n))$ es irreducible, entonces la imagen de C es un elemento del centro de G. Como $\det C = \det\left(\prod[A_i, B_i]\right) = 1$ y $C = \xi_n Id$, donde ξ_n es una raíz de la unidad, esto da lugar a diferentes componentes que corresponden a las diferentes posibles elecciones de ξ_n. Esta correspondencia fue generalizada para variedades Kähler por Donaldson [20, 21], Mehta y Ramanathan [59], y Uhlenbeck y Yau [79].

Cuando $G = GL(n, \mathbb{C})$, las variedades de caracteres ordinarias y de tipo twisted han sido un objeto central de estudio en las últimas décadas. Muchos de los aspectos geométricos y algebraicos relevantes que aparecían en el caso compacto lo hacen de nuevo: la Teoría de Invariantes Geométricos es utilizada para la construcción del cociente y aparecen también en el caso twisted varias componentes conexas que corresponden a las raíces n-ésimas de la unidad. Cada una de ellas es una variedad algebraica compleja afín no singular de dimensión compleja $2(n^2(g-1)+1)$. El resultado principal en éste área se conoce como la *correspondencia de Hodge no abeliana*, y establece isomorfismos (de distinta naturaleza según el caso) entre los siguientes espacios de móduli:

- La variedad de $GL(n, \mathbb{C})$-caracteres $\mathcal{M}_B(GL(n, \mathbb{C}))$, también llamada espacio de móduli de Betti: el espacio de móduli de representaciones del grupo fundamental de Σ en $GL(n, \mathbb{C})$.

- El espacio de móduli de de Rham, $\mathcal{M}_{dR}(GL(n, \mathbb{C}))$, que parametriza $GL(n, \mathbb{C})$-fibrados planos sobre Σ.

- El espacio de móduli de Dolbeault, $\mathcal{M}_{Dol}(GL(n, \mathbb{C}))$, el espacio de móduli de $GL(n, \mathbb{C})$-fibrados de Higgs sobre Σ.

La correspondencia puede verse como el análogo para el caso no abeliano a los isomorfismos

$$\operatorname{Hom}(\pi_1(\Sigma), \mathbb{C}) \cong H^1_B(X, \mathbb{C}) \cong H^1_{dR}(X, \mathbb{C})$$

existentes entre los distintos grupos de cohomología de una variedad compacta Kähler. La correspondencia de Riemann-Hilbert proporciona el vínculo entre los espacios $\mathcal{M}_B(G)$ y $\mathcal{M}_{dR}(G)$ [77, 16]. Asocia un $GL(n, \mathbb{C})$-fibrado plano a cada representación del grupo fundamental y viceversa, y da lugar a un isomorfismo analítico complejo entre ambos espacios.

En el otro lado de la correspondencia, un fibrado de Higgs es un par (E, Φ), donde E es un fibrado holomorfo y Φ, el campo de Higgs, es una sección holomorfa de $\mathrm{End}\,(E) \otimes K$, donde K es el fibrado canónico de Σ. Los fibrados de Higgs fueron introducidos por Hitchin [47] y contribuciones por parte de numerosos matemáticos ([22, 13, 50, 74] condujeron a Simpson [76] a la correspondencia final arriba mencionada entre los tres espacios. La relación entre $\mathcal{M}_{Dol}(G)$ y $\mathcal{M}_{dR}(G)$ puede describirse mediante la teoría de fibrados armónicos. Si partimos de una conexión plana ∇, ésta puede ser escrita localmente como $\nabla = d + A$, con $A \in \Omega^1(\Sigma, \mathrm{End}\ E)$. En presencia de una métrica hermítica h, ∇ descompone como $\nabla = d_A + \phi$, donde d_A es una $U(n)$-conexión y ϕ es un elemento de $\Omega^1(\Sigma, \mathrm{End}\,(h, E))$, el fibrado de endomorfismos hermíticos de E. Las ecuaciones para que la conexión A sea plana se convierten en el conjunto de ecuaciones:

$$\begin{aligned} F_A + \frac{1}{2}\,[\phi, \phi] &= 0 \\ d_A \phi &= 0 \end{aligned} \tag{0.1}$$

Si además se requiere que la métrica h sea armónica, aparece una ecuación adicional dada por $d_A^* \phi = 0$. Nótese que d_A define una estructura holomorfa en E. Resultados de Donaldson y Corlette establecen una correspondencia entre clases de equivalencia de $GL(n, \mathbb{C})$-fibrados planos y clases de equivalencia de fibrados armónicos: pares (d_A, ϕ) que satisfacen (0.1). El grupo gauge complejo actúa en el primer espacio, mientras que el grupo gauge hermítico es el que actúa en el segundo. Del fibrado armónico, utilizando la estructura compleja de Σ, podemos obtener descomposiciones $d_A = \bar{\partial}_A + d_A^{(1,0)}$ y $\psi = \Phi - \tau(\Phi)$, donde $\Phi \in \Omega^{1,0}(\Sigma, \mathrm{End}\ E)$ y $\tau(a) = -a^*$. El conjunto de ecuaciones (0.1) pueden reescribirse como

$$\begin{aligned} \bar{\partial}_A \Phi = [\Phi, \Phi] &= 0 \\ F_A - [\Phi, \tau(\Phi)] &= 0 \end{aligned} \tag{0.2}$$

que afirman precisamente que el par $(\overline{\partial}_A, \Phi)$ es un fibrado de Hiigs que satisface una ecuación extra proveniente de la armonicidad de h. En términos de fibrados de Higgs, esta condición es equivalente a su poliestabilidad. La afirmación precisa es que existe un homeomorfismo entre el espacio de móduli de clases de equivalencia de fibrados de Higgs poliestables y el espacio de conexiones planas reductivas modulo el grupo gauge complejo. Hay incontables aplicaciones y consecuencias que se han derivado de esta correspondencia de Hodge no abeliana. Si nos centramos en la topología de estos espacios de móduli, los tres son homeomorfos, luego comparten los mismos números de Betti y el mismo polinomio de Poincaré. Éste ha sido calculado principalmente desde el punto de vista del espacio de móduli de Dolbeault, por Hitchin [47] para $G = SL(2, \mathbb{C})$, por Gothen para rango 3 ($G = SL(3, \mathbb{C})$, [31]) y, recientemente, una fórmula recursiva para grado y rango arbitrario coprimo ha sido dada utilizando motivos [27].

Mirror symmetry y variedades de caracteres

Un aspecto destacable del espacio de móduli de fibrados de Higgs fue introducido por Hitchin en [48]: el espacio de móduli de Dolbeault, $\mathcal{M}_{Dol}$, es un sistema algebraico completamente integrable. Damos aquí una visión esquemática de la construcción e introducimos algunas de sus ideas más relevantes.

Podemos considerar $\mathcal{N} := \mathcal{N}(n,d)$, el espacio de móduli de fibrados vectoriales estables de grado d y rango n dentro de $\mathcal{M}_{Dol}$, si tomamos el campo de Higgs igual a cero. Si miramos al fibrado cotangente de este espacio, $T^*\mathcal{N}$, gracias a la dualidad de Serre y a teoría de deformación, tenemos una identificación

$$T^*\mathcal{N} \cong H^0(\Sigma, \mathrm{End}\,(E) \otimes K)$$

que permite interpretar en este contexto el campo de Higgs como un vector cotangente. De hecho, $\mathcal{M}_{Dol}$ es una variedad algebraica cuasi-proyectiva no singular que tiene a $T^*\mathcal{N}$ como un subconjunto abierto de Zariski. El campo de Higgs puede ser visto como una matriz de 1-formas, por lo que es natural considerar su polinomio característico, $\det(\Phi - tI) = t^n + a_0 t^{n-1} + \ldots + a_n$, donde $a_i \in H^0(\Sigma, K^i)$. Esta idea da lugar a la conocida como fibración de Hitchin, dada por la aplicación

$$\begin{aligned} h : \mathcal{M}_{Dol} &\longrightarrow \mathcal{A} := \bigoplus_{i=0}^{n} H^0(\Sigma, K^i) \\ (E, \Phi) &\longrightarrow (a_0, \ldots, a_n) \end{aligned}$$

Obsérvese que $\{a_i = tr(\Phi^i)\}$ es una base homogénea para el álgebra de polinomios invariantes de $\mathfrak{gl}(n, \mathbb{C})$. Además, se tiene que $\mathcal{A} = \dim \mathcal{M}/2$ y que h es una aplicación propia. Las fibras genéricas de h son abiertos de variedades abelianas: si tomamos un punto en la base de Hitchin, $a \in \mathcal{A}$, la ecuación dada por el polinomio característico define una curva S_a dentro del espacio total del fibrado canónico K que proyecta a Σ, $S_a \to \Sigma$. S_a es conocida como la *curva espectral*, y es lisa e irreducible para un punto genérico en la base de Hitchin. Existe una correspondencia entre fibrados de línea sobre S_a y fibrados de Higgs sobre Σ, vía imagen directa; es este hecho el que permite identificar la fibra de la fibración de Hitchin con la variedad de Picard de fibrados de línea sobre S_a de rango adecuado. Para otros grupos G existe una fibración de Hitchin $\mathcal{M}_{Dol}(G)$, aunque con algunas peculiaridades (por ejemplo, para $G = SL(n, \mathbb{C})$ la fibra genérica es biholomorfa a la variedad de Prym de la curva espectral).

En este contexto, podemos considerar la fibración de Hitchin para $G = SL(n, \mathbb{C})$ y su dual de Langlands, $PGL(n, \mathbb{C})$. Se tiene el siguiente diagrama,

$$\begin{tikzcd}
\mathcal{M}_{Dol}(SL(n,\mathbb{C})) \arrow[dr, "h"] & & \mathcal{M}_{Dol}(PGL(n,\mathbb{C})) \arrow[dl, "h"'] \\
& \mathcal{A}(SL(n,\mathbb{C})) \cong \mathcal{A}(PGL(n,\mathbb{C})) &
\end{tikzcd}$$

donde las fibras genéricas son variedades abelianas duales [38, 19]. Si cambiamos la estructura compleja de estos espacios y miramos a los espacios de móduli de De Rham, la fibración de Hitchin resulta ser una fibración lagrangiana y $\mathcal{M}_{DR}$ satisfacen las condiciones de Strominger-Yau-Zaslow de simetría mirror para variedades Calabi-Yau [78]. Estas consideraciones dieron lugar al estudio de los números de Hodge de $\mathcal{M}_{Dol}, \mathcal{M}_{DR}$ y $\mathcal{M}_B$, para confirmar ciertas conjeturas de simetra mirror topológica que haban sido verificadas para rango bajo y que predicen que ciertos números de Hodge de $\mathcal{M}_{dR}(G)$ y $\mathcal{M}_{dR}(G^L)$ coinciden. En particular, varias conjeturas y resultados que involucran a los E-polinomios de estos espacios (polinomios que se definen a partir de los números de Hodge) fueron dadas en [37]. Ésta es la motivación y el punto de partida del trabajo desarrollado en esta tesis.

E-polinomios

Cuando X es una variedad compacta Kähler, un resultado clásico acerca de su cohomología es la llamada *descomposición de Hodge*. Afirma que existe una descomposición en suma directa

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}$$

tal que $\overline{H^{p,q}} = H^{q,p}$. Los subespacios complejos $H^{p,q}$ tienen interpretación en teoría de haces, en términos de la conocida como cohomología de Dolbeault. Pueden ser caracterizados como el espacio de aquellas clases de cohomología de de Rham que pueden ser representadas por formas diferenciales complejas de tipo (p, q). Una generalización de este hecho fue dada por Deligne [17, 18] para variedades algebraicas cuasi-proyectivas, como son las variedades de caracteres. En ese caso, para cada k, los grupos de cohomología de X con coeficientes complejos $H^k(X, \mathbb{C})$ pueden ser dotados de dos filtraciones, una ascendente, *de pesos*

$$0 = W_{-1} \subseteq W_0 \subseteq \ldots \subseteq W_{2k} = H^{2k}(X, \mathbb{C})$$

y otra descendente, *la filtración de Hodge*,

$$H^k(X, \mathbb{C}) = F^0 \supseteq F^1 \supseteq \ldots \supseteq F^{n+1} = 0$$

que permiten definir los *números de Hodge mixtos* $h^{k,p,q}$ a partir de las piezas graduadas,

$$h^{k,p,q} = dim_{\mathbb{C}}(Gr_F^p Gr_{p+q}^W H^k(X, \mathbb{C})).$$

Este tipo de estructura es conocida como *estructura de Hodge mixta*, en contraposición con la estructura de Hodge *pura*, que se da cuando la filtración de pesos es trivial, $0 = W_{-1} \subseteq W_k = H^k(X, \mathbb{C})$, como sucede en el caso compacto Kähler.

Si miramos a los grupos de cohomología de los espacios de móduli que aparecen en la correspondencia de Hodge no abeliana, las estructuras de Hodge mixtas coinciden en $\mathcal{M}_{Dol}$ y $\mathcal{M}_{dR}$ y son de hecho puras [34], dada la existencia de compactificaciones lisas para ambos espacios. No obstante, la estructura de Hodge mixta de $H^*(\mathcal{M}_B)$ no es pura, por lo que el espacio de móduli de Betti no puede ser isomorfo como variedad algebraica a los espacios previous, incluso pese a que el isomorfismo de Riemann-Hilbert que lleva una conexión plana a su holonomía proporciona un isomorfismo analítico entre ambos espacios. Por lo tanto, existen dos filtraciones de pesos para $H^k(\mathcal{M}_{Dol}, \mathbb{C})$: una proveniente de la estructura de Hodge pura de $\mathcal{M}_{Dol}$ y otra obtenida de la estructura de Hodge mixta de $\mathcal{M}_B$ mediante la correspondencia de Hodge no abeliana. En [15], de Cataldo, Hausel and Migliorini proporcionaron una explicación de este hecho probando que la segunda filtración coincide con la filtración de Leray perversa asociada a la fibración de Hitchin $\mathcal{M}_{Dol} \to \mathcal{A}$ para rango 2.

Con la motivación de la simetría mirror y por las consideraciones previas, los E-polinomios de estos espacios fueron objeto de estudio. Partiendo de los números de Hodge mixtos con soporte compacto $h_c^{k,p,q}$ de X, el E-polinomio se define tomando características de Euler,

$$\chi_c^{p,q} := \sum_k (-1)^k h_c^{k,p,q}$$

y formando el polinomio

$$e(u,v) := \sum_{p,q} \chi_c^{p,q} u^p v^q$$

Nótese que cuando la estructura de Hodge mixta es pura, pueden recuperarse todos los números de Hodge mixtos a partir del E-polinomio. En [38], se realizó el cálculo de estos polinomios para el espacio de móduli de Dolbeault para $SL(n, \mathbb{C})$ y $PGL(n, \mathbb{C})$, $n = 2, 3$, donde se conjeturó también que de hecho coinciden para todo $n \in \mathbb{N}$. La afirmación exacta de esta conjetura involucra los llamados E-polynomios *stringy*, una versión que tiene en cuenta las singularidades de estos espacios, pero que coincide con los E-polinomios ordinarios en el caso liso.

Los E-polinomios de variedades de caracteres fueron analizados por Hausel y Rodriguez-Villegas en [37]. En su artículo, calcularon los E-polinomios del espacio de móduli de Betti

$\mathcal{M}_B^d$ (en el caso twisted) para $GL(n, \mathbb{C})$ utilizando poderosas herramientas aritméticas. El primer ingrediente de su método es un teorema de Katz, que relaciona los E-polinomios con el número de puntos que la variedad tiene sobre cuerpos finitos. Si consideramos una variedad algebraica cuasi-proyectiva X definida sobre $\mathbb{Z}$, decimos que X es de *conteo polinomial* si el número de puntos de X sobre un cuerpo finito $\mathbb{F}_q$ viene dado por un polinomio $P(q)$. El teorema de Katz afirma que para estas variedades de conteo polinomial, podemos obtener sus E-polinomios a partir de P, como

$$e(u, v) = P(uv) := \#\{X(\mathbb{F}_q)\}$$

donde $q = uv$. El resultado es especialmente útil para aquellas variedades cuya estructura de Hodge mixta es de tipo Hodge-Tate, como es el caso para nuestras variedades de caracteres ($h^{k,p,q} = 0$ para $p + q \neq k$, en ocasiones esta condición se llama de tipo equilibrado). El segundo ingrediente es calcular el número de puntos de las variedades de caracteres sobre cuerpos finitos, utilizando aritmética y la fórmula de caracteres

$$P(q) = \#(\mathcal{M}_B^d(\mathbb{F}_q)) = \sum_{\chi \in Irr(GL(n,\mathbb{F}_q))} \frac{|GL(n, \mathbb{F}_q)|^{2g-2}}{\chi(I_n)^{2g-1}} \chi(\xi_n I_n)$$

donde χ recorre todos los caracteres irreducibles. El cálculo final involucra las tablas de caracteres de $GL(n, \mathbb{F}_q)$ y proporciona una función generatriz para los E-polinomios. Para $n = 2$, utilizando la descripción explícita del anillo de cohomología de $\mathcal{M}_{Dol}$ dado en [40, 39], fueron capaces de calcular su polinomio de Hodge mixto, que recoge todos los números $h_c^{k,p,q}$. Este trabajo dio lugar a numerosas conjeturas, como fórmulas generales para los polinomios de Hodge mixtos de estos espacios o también relaciones de tipo cohomológico, tales como una dualidad de Poincaré "curiosa" o un teorema de Lefschetz duro también "curioso". Merece la pena también destacar la Conjetura de pureza, que relaciona la parte pura de la estructura de Hodge mixta con el A-polinomio de ciertos carcajs (quivers).

Más resultados en esta dirección fueron obtenidos más tarde por Hausel, Letellier y Rodriguez-Villegas en una serie de artículos [35, 36], para $GL(n, \mathbb{C})$-variedades de caracteres asociadas a superficies de Riemann con agujeros, con monodromías semisimples genéricas en los mismos. Obtuvieron de nuevo una función generatriz para los E-polinomios que involucraba polinomios de McDonald en este caso. Asimismo, fueron probadas más conexiones con la teoría de representaciones de carcajs y con esquemas de Hilbert. Para $SL(n, \mathbb{C})$, Mereb obtuvo también una función generatriz para los E-polinomios de variedades de caracteres en el caso twisted [60].

Método geométrico

En [53], Logares, Muñoz and Newstead proporcionaron un método alternativo para el cálculo de los E-polinomios de las variedades de caracteres. En este caso, optaron por un enfoque geométrico, basado en el estudio del comportamiento de los E-polinomios en fibraciones localmente triviales en la topología analítica. Su estudio utilizó estratificaciones adecuadas de estos espacios, útiles por el hecho de que los E-polinomios son aditivos con respecto a ellas. Los resultados principales en [53] son fórmulas explícitas para los E-polinomios de $SL(2,\mathbb{C})$-variedades de caracteres asociadas a superficies de género bajo ($g = 1, 2$) con un agujero. Sus técnicas permiten tratar con casos donde la monodromía en los agujeros es de tipo arbitrario, incluso no semisimple (caso en el que no hay correspondencia con ningún espacio de móduli de Higgs). Cuando la monodromía sí es de tipo semisimple, el espacio de móduli de Betti correspondiente es de tipo *parabólico*, homeomorfo al espacio de móduli de fibrados de Higgs con estructuras parabólicas sobre los agujeros [75]. El comportamiento de estos espacios $\mathcal{M}_\lambda^{g=1}(\lambda \neq 0, \pm 1)$ para género 1 a medida que λ varía está codificado en la llamada *representación de la monodromía de Hodge*, una manera adecuada de capturar la información de los E-polinomios de la fibra y del espacio total de la fibración dada por la familia $\mathcal{M}_\lambda \to \mathbb{C} - \{0, \pm 1\}$. Es este punto de vista geométrico el adoptado y desarrollado a lo largo de esta tesis doctoral.

Objetivos

El principal objetivo de la presente tesis es extender el estudio geométrico de los E-polinomios de variedades de caracteres asociadas a curvas complejas de género bajo descrito en [53]. La primera meta es aplicar las técnicas que aparecen en dicho artículo a otras $SL(2,\mathbb{C})$-variedades de caracteres, con el fin de obtener descripciones explícitas de estos espacios que permitan obtener sus E-polinomios y obtener así nueva información topológica. El segundo objetivo principal es estudiar el caso de la variedad de caracteres asociada a una curva de género 3 para $G = SL(2,\mathbb{C})$. Es claro en un primer intento que las herramientas descritas en [53] no son suficientes y que necesitan ser extendidas para fibraciones donde la base es de dimensión mayor que uno, por lo que otra tarea es proporcionar la maquinaria y el contexto necesario para resolver el problema, ya que la complejidad de los espacios estudiados crece enormemente a medida que aumenta el género. Además, las fórmulas existentes en la literatura cubren tan sólo algunos casos y utilizan métodos aritméticos basados en contar puntos sobre cuerpos finitos, por lo que sería deseable hallar cualquier relación entre los E-polinomios y la geometría de estos espacios. El tercer y último objetivo es obtener fórmulas cerradas y explícitas para género y monodromía arbitrarias, para arrojar luz en el problema y confirmar ciertos fenómenos ya presentes para género bajo.

Resultados

El Capítulo 1 introduce las variedades de caracteres y define los conceptos básicos necesarios para el resto de secciones. Se centra no obstante en un problema particular como ejemplo que sirve a la vez de guía, las $SU(2)$-variedades de caracteres asociadas a grupos de nudos tóricos. Estos grupos admiten la siguiente presentación

$$G_{m,n} = \langle x, y \mid x^m = y^n \rangle,$$

donde $m, n \in \mathbb{N}$ son coprimos. Las variedades de caracteres de estos grupos, en el caso $(m, 2)$ y para $G = SL(2, \mathbb{C}), SU(2)$ fueron tratadas en [64] y [66] mediante herramientas combinatorias. Para (m, n) arbitrarios y $G = SL(2, \mathbb{C})$, fueron geométricamente descritas en [61] utilizando caracteres. Cabe destacar que las variedades de caracteres asociadas a otros grupos de nudos han sido estudiadas por varios autores, como método para producir invariantes asociados a éstos [8, 42, 43, 44, 45].

El lugar geométrico dado por las representaciones irreducibles en $\mathcal{M}_{SL(2,\mathbb{C})}(G_{m,n})$ es una colección de $\frac{(m-1)(n-1)}{2}$ rectas complejas cuya clausura interseca a la parte reducible, isomorfa a $\mathbb{C}$. En este capítulo, analizamos la inyección

$$i_* : \mathcal{M}_{G_{m,n}}(SU(2)) \longrightarrow \mathcal{M}_{G_{m,n}}(SL(2, \mathbb{C}))$$

El conjunto de clases de equivalencia de representaciones reducibles en $\mathcal{M}_{G_{m,n}}(SU(2))$ es isomorfo a un intervalo real cerrado, mientras que el conjunto dado por las irreducibles consta de $\frac{(m-1)(n-1)}{2}$ intervalos reales abiertos dentro de sus correspondientes rectas en $\mathcal{M}^{irr}_{G_{m,n}}(SL(2, \mathbb{C}))$. La descripción es análoga para $m, n \in \mathbb{N}$ cualesquiera. A partir de la descripción geométrica, se obtiene el siguiente corolario

Corolario 1.3.5. *$\mathcal{M}_{G_{m,n}}(SU(2))$ es un retracto de deformación de $\mathcal{M}_{G_{m,n}}(SL(2, \mathbb{C}))$.*

Se sabe que dado un grupo de Lie complejo G y $K \subset G$ un subgrupo compacto maximal, hay ciertos grupos para los cuales la K-variedad de caracteres es un retracto de deformación de la G-variedad de caracteres [24, 26], pero que hay otros para los que no [5]. La caracterización precisa de aquellos grupos para los que la cuestión es afirmativa es todavía un problema abierto.

El segundo capítulo de esta disertación introduce la teoría de Hodge de variedades algebraicas y define y prueba las propiedades básicas que cumplen los E-polinomios. En particular, se proporcionan las herramientas principales para el manejo de fibraciones $F \to$

$E \to B$ que son localmente triviales para la topología analítica y cuya estructura de Hodge mixta es de tipo Hodge-Tate. Cuando la monodromía es finita y abeliana (es decir, cuando factoriza a través de un grupo finito Γ de estas características), cada $H^{k,p,p}(F)$ puede ser visto como un módulo sobre el anillo de representaciones del grupo, $R(\Gamma)$. La representación de la monodromía de Hodge es un polinomio con coeficientes en el anillo de representaciones de Γ,

$$R(E) := \sum (-1)^k H_c^{k,p,p}(F) q^p \in R(\Gamma)[q]$$

que captura tanto la información de los E-polinomios de la base y la fibra como la dada por la monodromía de la fibración. El capítulo 2 extiende los resultados dados en [53] para fibraciones con base 1-dimensional (Corolario 2.3.5) a bases de dimensión superior (Teorema 2.3.2, Corolario 2.3.7), necesarias en el Capítulo 4 para calcular los E-polinomios de variedades de caracteres asociadas a superficies de Riemann de género 3. Además, el Teorema 2.3.2 proporciona el marco adecuado para atacar en un futuro el problema para rango arbitrario.

Como se ha mencionado, los E-polinomios de variedades de caracteres de curvas complejas de género 1, así como sus representaciones de la monodromía de Hodge, fueron calculadas en [53]. En esta tesis se hace repetido uso de ellas, y dado que son necesarias para solucionar el problema para género arbitrario, las llamamos piezas básicas. Para obtener sus E-polinomios, es necesario manejar las ecuaciones explícitamente y obtener representantes para la acción por conjugación, así como estratificar y estudiar la monodromía caso a caso. En el capítulo 3, ilustramos esta técnica resolviendo un problema similar, el de variedades de caracteres asociadas a superficies no orientables de género bajo. Han sido estudiadas recientemente desde el punto de vista de los fibrados de Higgs y se ha mostrado que una correspondencia de tipo Hodge no abeliana es también cierta [49]. En esta tesis, se calculan sus polinomios de Hodge y se extrae información topológica a partir de ellos. El resultado principal del Capítulo 3 son los siguientes teoremas

Teorema 3.1.1. *Sea K la botella de Klein. Los E-polinomios de las $SL(2,\mathbb{C})$-variedades de caracteres $\mathcal{M}_C(K)$ vienen dados por*

$$\begin{aligned} e(\mathcal{M}_{\mathrm{Id}}(K)) &= 3q-2, \\ e(\mathcal{M}_{-\mathrm{Id}}(K)) &= q-1, \\ e(\mathcal{M}_{J_+}(K)) &= q^2+2q-7, \\ e(\mathcal{M}_{J_-}(K)) &= q^2+3q, \\ e(\mathcal{M}_{\xi_\lambda}(K)) &= q^2+2q+1. \end{aligned}$$

Teorema 3.1.2. *Sea Σ la suma conexa de tres planos proyectivos. Los E-polinomios de sus $SL(2,\mathbb{C})$-variedad de caracteres asociadas son*

$$\begin{aligned}
e(\mathcal{M}_{\mathrm{Id}}(\Sigma)) &= q^3 - 6q - 1, \\
e(\mathcal{M}_{-\mathrm{Id}}(\Sigma)) &= 2q^3 + 7q^2 - 1, \\
e(\mathcal{M}_{J_+}(\Sigma)) &= q^5 + 5q^3 + 12q^2 - 8q + 26, \\
e(\mathcal{M}_{J_-}(\Sigma)) &= q^5 - 5q^3 - 12q^2, \\
e(\mathcal{M}_{\xi_\lambda}(\Sigma)) &= q^5 + q^4 + 2q^3 + 8q^2 - 27q + 23.
\end{aligned}$$

Los capítulos 4 y 5 pueden ser considerados la parte central de esta tesis doctoral y en ellos se desarrolla el caso de $SL(2,\mathbb{C})$-variedades de caracteres de curvas complejas de género $g \geq 3$. En el capítulo 4 se calculan los E-polinomios de la variedad de caracteres ordinaria $\mathcal{M}_{\mathrm{Id}}$ y de la variedad de caracteres twisted $\mathcal{M}_{-\mathrm{Id}}$ para género $g = 3$. Para acometer tal tarea, se estratifica de modo conveniente el espacio de representaciones en $SL(2,\mathbb{C})^6$ y es en este punto donde las fibraciones con bases de dimensión mayor que uno aparecen. El caso $g = 3$ sirve como la base del proceso inductivo llevado a cabo para género arbitrario en el capítulo 5. La idea básica es la descomposición de X^g como suma conexa $X^g = X^{g-1}\#X^1$. A partir de X^g, se obtiene información de X^{g-1} con un agujero, lo que es utilizado para calcular el polinomio correspondiente a $X^{g+1} = X^{g-1}\#X^2$. En este punto se utilizan tanto los bloques básicos para $g = 1, 2$ como el caso de género 3 y las representaciones de las monodromías de Hodge para género 2 calculadas en el Capítulo 4.

Los resultados principales obtenidos son:

Teorema 5.1.1. *Sea X una curva compleja de género $g \geq 1$. Sea $\mathcal{M}_C^g = \mathcal{M}_C^g(SL(2,\mathbb{C}))$ la variedad de caracteres correspondiente a $C \in SL(2,\mathbb{C})$. Los E-polinomios de $\mathcal{M}_C^g$ son*

$$\begin{aligned}
e(\mathcal{M}_{\mathrm{Id}}^g) &= (q^3-q)^{2g-2} + (q^2-1)^{2g-2} - q(q^2-q)^{2g-2} - 2^{2g}q^{2g-2} \\
&\quad + \frac{1}{2}q^{2g-2}(q + 2^{2g} - 1)((q+1)^{2g-2} + (q-1)^{2g-2}) \\
&\quad + \frac{1}{2}q((q+1)^{2g-1} + (q-1)^{2g-1}). \\
e(\mathcal{M}_{-\mathrm{Id}}^g) &= (q^3-q)^{2g-2} + (q^2-1)^{2g-2} - 2^{2g-1}(q^2+q)^{2g-2} + (2^{2g-1}-1)(q^2-q)^{2g-2}. \\
e(\mathcal{M}_{J_+}^g) &= (q^3-q)^{2g-2}(q^2-1) + (2^{2g-1}-1)(q-1)(q^2-q)^{2g-2} \\
&\quad - 2^{2g-1}(q+1)(q^2+q)^{2g-2} + \frac{1}{2}q^{2g-2}(q-1)\left((q-1)^{2g-1} - (q+1)^{2g-1}\right). \\
e(\mathcal{M}_{J_-}^g) &= (q^3-q)^{2g-2}(q^2-1) \\
&\quad + (2^{2g-1}-1)(q-1)(q^2-q)^{2g-2} + 2^{2g-1}(q+1)(q^2+q)^{2g-2}. \\
e(\mathcal{M}_{\xi_\lambda}^g) &= (q^3-q)^{2g-2}(q^2+q) + (q^2-1)^{2g-2}(q+1) + (2^{2g}-2)(q^2-q)^{2g-2}q,
\end{aligned}$$

donde $J_+ = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $J_- = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$ *y* $\xi_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, $\lambda \neq 0, \pm 1$, *con* $q = uv$.

Teorema 5.1.2. *Todas las variedades de caracteres $\mathcal{M}_C(SL(2,\mathbb{C}))$ son de tipo Hodge-Tate (tipo equilibrado).*

El comportamiento de la cohomología de las variedades de caracteres parabólicas para género arbitrario queda recogido en su representación de la monodromía de Hodge,

Teorema 5.1.4. *Sea X una curva compleja de género $g \geq 1$. Entonces su representación de la monodromía de Hodge viene dada por*

$$\begin{aligned} R(\mathcal{M}^g_{\xi_\lambda}) = &\left((q^3-q)^{2g-2}(q^2+q) + (q+1)(q^2-1)^{2g-2} - q(q^2-q)^{2g-2}\right) T \\ &+ \left((2^{2g}-1)q(q^2-q)^{2g-2}\right) N, \end{aligned}$$

donde el E-polinomio de la parte invariante de la cohomología es el coeficiente que acompaa a T, mientras que la parte no invariante es el polinomio que acompaña a N, donde T, N son la representación trivial y no trivial respectivamente.

Por último, deducimos algunas implicaciones del Teorema 5.1.1.

Corolario 5.9.1. *Sea X una curva compleja de género $g \geq 2$. La característica de Euler de $\mathcal{M}^g_C = \mathcal{M}_C(SL(2,\mathbb{C}))$ viene dada por*

$$\begin{aligned} \chi(\mathcal{M}^g_{\mathrm{Id}}) &= 2^{4g-3} - 3 \cdot 2^{2g-2}, \\ \chi(\mathcal{M}^g_{-\mathrm{Id}}) &= -2^{4g-3}, \\ \chi(\mathcal{M}^g_{J_+}) &= -2^{4g-2}, \\ \chi(\mathcal{M}^g_{J_-}) &= 2^{4g-2}, \\ \chi(\mathcal{M}^g_{\xi_\lambda}) &= 0. \end{aligned}$$

Corolario 5.9.2. *Sea X una curva compleja de género $g \geq 2$. Entonces $\mathcal{M}^g_{\mathrm{Id}}$ y $\mathcal{M}^g_{-\mathrm{Id}}$ son de dimensión $6g-6$ y $\mathcal{M}^g_{J_+}$, $\mathcal{M}^g_{J_-}$ y $\mathcal{M}^g_{\xi_\lambda}$ son de dimensión $6g-4$. Todas ellas tienen una única componente de dimensión máxima.*

Corolario 5.9.3. *Sea X una curva compleja de género $g \geq 1$. Entonces $e(\mathcal{M}^g_{-\mathrm{Id}})$, $e(\mathcal{M}^g_{\xi_\lambda})$, y sus partes invariantes y no invariantes dadas en el Teorema 4.3 son polinomios palíndromos.*

Conclusiones

El estudio geométrico de los E-polinomios asociados a $SL(2,\mathbb{C})$-variedades de caracteres asociadas a curvas complejas de género arbitrario ha sido llevado a cabo con éxito en esta tesis. Los Teoremas 5.1.1, 5.1.2 y 5.1.4 son los principales resultados de esta disertación y cumplen los objetivos que fueron marcados en un principio. El primero de ellos se consigue mediante el estudio de las variedades de caracteres asociadas a superficies no orientables de género bajo, donde las técnicas establecidas en [53] funcionan. Otro problema interesante que involucra $SL(2,\mathbb{C})$-variedades de caracteres, asociadas a nudos tóricos, es estudiado en el Capítulo 1. En él, la descripción geométrica en términos de caracteres da una respuesta afirmativa al problema de cuándo la $SU(2)$-variedad de caracteres es un retracto de la correspondiente $SL(2,\mathbb{C})$-variedad de caracteres. Aunque el problema general sigue abierto, proporciona un ejemplo nuevo que no es abeliano ni libre.

El caso de género 3 se resuelve estudiando fibraciones sobre bases de dimensión 2, y éstas resultan suficientes también para solucionar el caso de género arbitrario. Pese a ello, el Teorema 2.3.2 funciona para cualquier fibración con monodromía abeliana y finita, sin restricciones dimensionales, por lo que es de esperar que sea útil en otros contextos. El objetivo final y principal de la tesis es desarrollado en el Capítulo 5, donde se proporcionan las fórmulas para género y monodromía arbitraria. Generalizan las dadas en [53] para $g = 1, 2$. El proceso de inducción proporciona además una novedad: la información de los E-polinomios de la variedad de caracteres para género g está codificado en los ocho polinomios $(e_0^g, e_1^g, e_2^g, e_3^g, a_g, b_g, c_g, d_g)$ (los cuatro primeros proporcionan la información de las monodromías Id, $-\,\mathrm{Id}$, J_+ y J_-, y los cuatro últimos vienen dados por el caso parabólico, y aparecen en la representación de la monodromía de Hodge). La información para género $g+1$ se obtiene a partir de la de género g mediante una aplicación lineal, dada por una cierta matriz de E-polinomios. En otras palabras, el procedimiento topológico de pegado de un asa a X_g con el que se obtiene X_{g+1} es reproducido por esta aplicación lineal a nivel de E-polinomios.

Las ideas y herramientas utilizadas para obtener los resultados del Capítulo 5 pueden ser aplicadas a otros problemas. Como primer paso, ciertamente permite atacar el problema de $PGL(2,\mathbb{C})$-variedades de caracteres asociadas a curvas complejas de género arbitrario, así como ampliar los resultados del Capítulo 3 a género arbitrario también. Estos problemas serán objeto de trabajo futuro. Por otro lado, las técnicas del Capítulo 2 abren la puerta al estudio de otros grupos, como $G = SL(3,\mathbb{C})$. Aunque aparecen dificultades técnicas en el cálculo de los llamados bloques básicos, el procedimiento de inducción podría llevarse a cabo de nuevo para género arbitrario. Esto sería un importante paso para la resolución del problema para rango arbitrario.

Los resultados principales de esta tesis están recogidos en los preprints [56, 57, 58]. El primero de ellos contiene los resultados sobre $SU(2)$-variedades de caracteres de nudos tóricos que aparecen en el Capítulo 1, mientras que el segundo y el tercero recogen el caso de $SL(2,\mathbb{C})$-variedades de caracteres de curvas de género 3 y género arbitrario respectivamente.

Introduction

This phD thesis is devoted to the study of certain algebraic invariants, called E-polynomials, that can be associated to any quasi-projective variety X. These E-polynomials (also called Hodge-Deligne polynomials in the literature) are of cohomological nature and contain topological and arithmetic information of X. We focus on a particular type of algebraic varieties called character varieties. Given a finitely presented group Γ and a reductive group G, they are the moduli space of representations of Γ into G,

$$\mathcal{M}_\Gamma(G) := \mathrm{Hom}\,(\Gamma, G)//G$$

where G acts by conjugation. The topological quotient of the space of representations by the G-action may not have nice properties, so some orbits may be identified and in this algebraic setting the moduli space is constructed using Geometric Invariant Theory. Character varieties are central objects in many branches of mathematics: they appear in dynamics, representation theory, algebraic and differential geometry... In this introduction we focus on certain aspects that are relevant to introduce and motivate the problem developed in this thesis, as well as to place it in its right context. See [71] and the references therein for a broader picture.

Geometric structures

The work of Felix Klein in the XIX century stablished that classical geometries should be regarded as properties of a space that are invariant under a transitive and continuous action (the action of a Lie group). Spaces like $S^n, \mathbb{C}P^n$... can be regarded as homogeneous spaces: quotients of their transitive group of isometries under a certain subgroup, the stabilizer of a point. A less rigid and natural question is to consider its local version. Given a fixed topological space Σ, one can wonder if it admits a local system of coordinates modeled on a fixed homogeneous space $X = G/H$ such that the changes of coordinates are isometries (elements of G). What geometries can be given to a particular space? These questions correspond to the notion of a (G, X)-structure on Σ.

Character varieties arise as moduli spaces that parametrize equivalence classes of these structures. Given an atlas $\{(U_\alpha, \psi_\alpha)\}_{\alpha\in\Lambda}$, where $\mathcal{U} = \{U_\alpha\}$ is an open cover of Σ and

$\psi_\alpha : U_\alpha \to X$ are local charts, we can take the coordinate changes over connected subsets $g_{\alpha\beta}(U_\alpha, U_\beta) \in G$ that satisfy that $\psi_\alpha = g_{\alpha\beta} \circ \psi_\beta$. These $\{g_{\alpha\beta}\}$ define a flat bundle as follows: take the trivial bundle E_α over each U_α, $E_\alpha = U_\alpha \times X \to U_\alpha$, and identify over the intersections $(u, x) \sim (u, g_{\alpha\beta}x)$. This procedure defines a bundle $E \to \Sigma$ with fibre X and structure group G. Note that the bundle pullbacks to the trivial bundle over the universal covering $\tilde{\Sigma}$, so it may be reconstructed from it. To be precise, we can recover E from $\tilde{\Sigma} \times X$ as its quotient by a π_1-action covering the action over $\tilde{\Sigma}$ by deck transformations. This action is determined by a homomorphism $\pi_1(\Sigma) \to G$, the *holonomy representation*. The key fact is that isomorphism classes of flat bundles correspond to elements of $\mathrm{Hom}\,(\pi_1(\Sigma), G)$ modulo inner automorphisms of G. In a different terminology, what we have just sketched is that there is a correspondence between G-local systems (flat G-bundles) and representations of the fundamental group of Σ into G.

Besides, the coordinate charts $\{\psi_\alpha : U_\alpha \to X\}$ glue to a global section of the bundle E, called the *developing section* in the literature. It can be regarded as a $\pi_1(X)$-equivariant map $\tilde{\Sigma} \xrightarrow{s} X$, where $\pi_1(\Sigma)$ acts on the latter via ρ. Varying G one obtains a different geometry, called a *locally homogeneous geometric structure* in this context. A classic example is when $G = Isom(\mathbb{H}) \cong PGL(2,\mathbb{R})$ is the group of isometries of the hyperbolic plane; the corresponding geometric structure will be hyperbolic. It is important here the Teichmuller space $\mathcal{T}(\Sigma)$, equivalent by the Uniformization Theorem to the set of equivalence classes of complete hyperbolic structures on Σ. In this setting, given a hyperbolic structure, the holonomy map gives a representation of $\pi_1(\Sigma)$ into $PGL(2,\mathbb{R})$, well defined up to conjugation. Hence there is a map

$$hol : \mathcal{T}(\Sigma) \longrightarrow \mathrm{Hom}\,(\Sigma, PGL(2,\mathbb{R}))/PGL(2,\mathbb{R}).$$

The map is an embedding and its image is the set of faithful and discrete representations. It corresponds to a connected component of the character variety. In general, there is a deformation space associated to marked (G,X)-structures and a holonomy map $hol : Def^{G,X}(\Sigma) \longrightarrow \mathrm{Hom}\,(\pi_1(\Sigma), G)//G$. For example, when $G = PSL(2,\mathbb{C})$ one recovers complex projective structures [23], or convex projective structures for $PSL(3,\mathbb{R})$ [12]. The study of topological invariants to distinguish and count connected components of these moduli spaces has been thoroughly treated [28, 9]. Of course, many other related questions that cover different areas have been studied over the years, such as dynamics, related to the action of the mapping class group [29, 10] or local rigidity of subgroups of a given Lie group [81, 70].

Moduli spaces of bundles. Non-abelian Hodge theory

The case $G = U(n)$ became relevant due to the work of Narasimhan and Seshadri [63], which grew out of the desire to understand holomorphic bundles on a Riemann surface. The cases where Σ is the Riemann sphere or an elliptic curve had already been studied by Grothendieck and Atiyah, respectively [33, 2]. In their paper of 1965, Narasimhan and Seshadri stablished that the moduli space of stable holomorphic vector bundles of rank n and degree 0 on a compact Riemann surface was diffeomorphic to the moduli space of irreducible representations of its fundamental group into $U(n)$, the $U(n)$-character variety of Σ. The topological structure of a complex vector bundle E over Σ is determined by its degree and its rank, and by the theorem, its differentiable structure is independent of the complex structure on Σ. Once a complex structure on Σ is fixed, the $(0,1)$-part of a unitary connection A defines a holomorphic structure on E: a section is said to be holomorphic if its annihilated by the $(0,1)$-part of the covariant derivative associated to A (recall that they are integrable since there are no $(0,2)$-forms on Σ). Since the connection is unitary, this $(0,1)$-part determines the whole A. Finally, the stability condition corresponds to the irreducibility of the associated unitary representation of the fundamental group and equivalent connections give conjugate representations. The moduli space can also be described as a symplectic quotient of the affine space of connections of the trivial vector bundle, where the moment map is given by the curvature map [3]. In the abelian case, the character variety is simply the complex torus of dimension g,

$$\mathrm{Hom}\,(\pi_1(\Sigma), U(1))/U(1) = \mathrm{Hom}\,(\pi_1(\Sigma), U(1)) \cong \mathrm{Hom}\,(H_1(\Sigma), U(1)) \cong U(1)^{2g},$$

which if we fix a complex structure on Σ, is precisely the moduli space of holomporhic line bundles on Σ, the Jacobian $\mathrm{Jac}(\Sigma) \cong \mathrm{Pic}^0(\Sigma)$.

For a vector bundle E of arbitrary degree d and rank n a similar correspondence holds, but since in that case E does not admit flat connections, stable holomorphic bundles over Σ correspond to projectively flat connections, which give representations of a central extension $\tilde{\pi}_1(\Sigma)$ of the fundamental group into $U(n)$. Explicitly, the corresponding twisted character variety is given by

$$\mathrm{Hom}\,(\tilde{\pi}_1(\Sigma), U(n))/U(n) = \Big\{(A_1, B_1, \ldots, A_g, B_g, C) \in U(n)^{2g+1} \mid \\ [A_i, C] = [B_i, C] = \mathrm{Id}, \prod_{i=1}^{g}[A_i, B_i] = C\Big\}$$

If $\rho \in \mathrm{Hom}\,(\tilde{\pi}_1(\Sigma), U(n))$ is irreducible, then C is mapped to a central element. Since $\det C = \det\left(\prod[A_i, B_i]\right) = 1$ and $C = \xi_n Id$, where ξ_n is a root of unity, this produces several components corresponding to the different choices of ξ_n. This correspondence was generalized to higher dimensional Kähler manifolds by Donaldson [20, 21], Mehta and Ramanathan [59], and Uhlenbeck and Yau [79].

When $G = GL(n, \mathbb{C})$, twisted and ordinary character varieties have been a central object of study in the last decades. Many of the algebraic and geometric features that were relevant in the compact case appear again: Geometric Invariant Theory is needed to consider an apropiate quotient and in the twisted case there are several components that correspond to the primitive n-th roots of unity. Each one of them is a non-singular affine algebraic variety of complex dimension $2(n^2(g-1)+1)$. The main result concerning these varieties is called the *non-abelian Hodge correspondence* and stablishes isomorphisms (of different nature) between the following moduli spaces:

- The $GL(n, \mathbb{C})$-character variety (called Betti moduli space) $\mathcal{M}_B(GL(n, \mathbb{C}))$: the moduli space of representations of the fundamental group of Σ into $GL(n, \mathbb{C})$.
- The De Rham moduli space, $\mathcal{M}_{dR}(GL(n, \mathbb{C}))$, the moduli space of $GL(n, \mathbb{C})$-flat bundles over Σ.
- The Dolbeault moduli space, $\mathcal{M}_{Dol}(GL(n, \mathbb{C}))$, the moduli space of $GL(n, \mathbb{C})$-Higgs bundles over Σ.

It can be seen as the non-abelian analogue of the standard isomorphisms

$$\mathrm{Hom}\,(\pi_1(\Sigma), \mathbb{C}) \cong H^1_B(X, \mathbb{C}) \cong H^1_{dR}(X, \mathbb{C})$$

between the different cohomology groups of a Kähler compact manifold. The Riemann-Hilbert correspondence provides the link between $\mathcal{M}_B(G)$ and $\mathcal{M}_{dR}(G)$ [77, 16]. It associates a flat $GL(n, \mathbb{C})$-bundle to each representation of the fundamental group and viceversa, and it sets a complex analytic isomorphism between both spaces.

On the other side of the picture, a Higgs bundle is a pair (E, Φ), where E is a holomorphic bundle and Φ, the Higgs field, is a holomorphic section of $\mathrm{End}\,(E) \otimes K$, where K is the canonical bundle of Σ. Higgs bundles were introduced by Hitchin [47] and contributions from many mathematicians ([22, 13, 50, 74]) led to Simpson's [76] final correspondence between the three spaces, stated above. The correspondence between $\mathcal{M}_{Dol}(G)$ and $\mathcal{M}_{dR}(G)$ is described using the theory of harmonic bundles. If we start with a flat connection ∇, it can be locally written as $\nabla = d + A$, where $A \in \Omega^1(\Sigma, \mathrm{End}\ E)$. In the presence of an hermitian metric h, ∇ decomposes further as $\nabla = d_A + \phi$, where d_A is a $U(n)$-connection and ϕ is an element in $\Omega^1(\Sigma, \mathrm{End}\,(h, E))$, the bundle of hermitian endomorphisms of E. The set of equations for the flatness of A turns into the set of equations

$$\begin{aligned} F_A + \frac{1}{2}\,[\phi, \phi] &= 0, \\ d_A \phi &= 0. \end{aligned} \tag{0.3}$$

If we require harmonicity for the metric h, we need to add an extra equation given by $d_A^* \phi = 0$. Note that d_A defines a holomorphic structure on E. Donaldson and Corlette's result stablishes a correspondence between equivalence classes of flat $GL(n, \mathbb{C})$-bundles and equivalence classes of harmonic bundles: pairs (d_A, ϕ) satisfying (0.3). Note that the complex gauge group acts on the first, whereas the hermitian gauge group acts on the latter. From the harmonic bundle, using the complex structure on Σ, we obtain decompositions $d_A = \bar{\partial}_A + d_A^{(1,0)}$ and $\psi = \Phi - \tau(\Phi)$, where $\Phi \in \Omega^{1,0}(\Sigma, \mathrm{End}\ E)$ and $\tau(a) = -a^*$. The set of equations (0.3) becomes

$$\begin{aligned} \bar{\partial}_A \Phi = [\Phi, \Phi] &= 0, \\ F_A - [\Phi, \tau(\Phi)] &= 0, \end{aligned} \tag{0.4}$$

which say that the pair $(\bar{\partial}_A, \Phi)$ is a Higgs bundle that satisfies an extra equation coming from the harmonicity of h. In terms of Higgs bundles, this condition is equivalent to polystability. The precise statement is that there is a homeomorphism between the moduli space of equivalence classes of polystable Higgs bundles and reductive flat connections modulo the complex gauge group. There have been countless consequences derived from this non-abelian Hodge correspondence. If we focus on the topology of these moduli spaces, the three of them are homeomorphic, so they all have the same Betti numbers and the same Poincare polynomial. They have been computed from the point of view of the Dolbeault moduli space, by Hitchin in [47] for $G = SL(2, \mathbb{C})$, by Gothen in the rank 3 case ($G = SL(3, \mathbb{C})$, [31]) and recently, a recursive formula has been given for arbitrary rank and coprime degree using motives [27].

Mirror symmetry and character varieties

A remarkable feature of the moduli space of Higgs bundles was introduced by Hitchin in [48]: the Dolbeault moduli space, $\mathcal{M}_{Dol}$, is an algebraically complete integrable system. We outline here the main points of the construction and introduce some of the relevant ideas.

We can consider $\mathcal{N} := \mathcal{N}(n, d)$, the moduli space of stable vector bundles of degree d and rank n inside $\mathcal{M}_{Dol}$, setting the Higgs field equal to zero. If we look at its cotangent bundle, $T^*\mathcal{N}$, using deformation theory and Serre duality we have an identification

$$T^*\mathcal{N} \cong H^0(\Sigma, End(E) \otimes K)$$

that allows us to interpret in this setting the Higgs field as a cotangent vector. In fact, $\mathcal{M}_{Dol}$ is a non-singular quasi-projective variety that has $T^*\mathcal{N}$ as a Zariski open subset. This Higgs field can be thought as a matrix of 1-forms, so it is natural to consider its characteristic polynomial, $\det(\Phi - tI) = t^n + a_0 t^{n-1} + \ldots + a_n$, where $a_i \in H^0(\Sigma, K^i)$. This idea leads to the Hitchin fibration, given by the map

$$\begin{aligned} h : \mathcal{M}_{Dol} &\longrightarrow \mathcal{A} := \bigoplus_{i=0}^{n} H^0(\Sigma, K^i) \\ (E, \Phi) &\longrightarrow (a_0, \ldots, a_n) \end{aligned}$$

Note that $\{a_i = tr(\Phi^i)\}$ is a homogeneous basis for the algebra of invariant polynomials of $\mathfrak{gl}(n, \mathbb{C})$. Besides, $\dim \mathcal{A} = \dim \mathcal{M}/2$ and h is a proper map. The generic fibres of h are open subsets of abelian varieties: if we pick a point in the Hitchin base, $a \in \mathcal{A}$, the equation given by the characteristic polynomial defines a curve S_a inside the total space of the canonical bundle K which maps onto Σ, $S_a \to \Sigma$. S_a is called the *spectral curve*, and it is smooth and irreducible for a generic point in the Hitchin base. There is a correspondence between line bundles over S_a and Higgs bundles over Σ, via direct image, and this is what identifies the fibre of the Hitchin map with the Picard variety of line bundles over S_a of suitable degree. For other groups G there is also a Hitchin map of $\mathcal{M}_{Dol}(G)$, although some modifications occur (for example, when $G = SL(n, \mathbb{C})$ the generic fibre is biholomorphically equivalent to the Prym variety of the spectral curve).

In this setup, we can consider the Hitchin map for $G = SL(n, \mathbb{C})$ and its Langlands dual, $PGL(n, \mathbb{C})$. There is a diagram

$$\begin{tikzcd} \mathcal{M}_{Dol}(SL(n,\mathbb{C})) \arrow[dr, "h"] & & \mathcal{M}_{Dol}(PGL(n,\mathbb{C})) \arrow[dl, "h"'] \\ & \mathcal{A}(SL(n,\mathbb{C})) \cong \mathcal{A}(PGL(n,\mathbb{C})) & \end{tikzcd}$$

such that the generic fibres are dual abelian varieties [38, 19]. Changing complex structures and looking at the De Rham moduli space, the Hitchin fibration becomes a special lagrangian fibration and $\mathcal{M}_{DR}$ satisfy the requirements of Strominger-Yau-Zaslow for mirror symmetric Calabi-Yau manifolds [78]. These considerations lead to the study of Hodge numbers of $\mathcal{M}_{Dol}, \mathcal{M}_{dR}$ and $\mathcal{M}_B$ to confirm a topological mirror symmetry conjecture that was verified for low rank cases that predicted that certain Hodge numbers of $\mathcal{M}_{dR}(G)$ and $\mathcal{M}_{dR}(G^L)$ agree. In particular, several conjectures and results involving the E-polynomials of these spaces (polynomials that are defined from these Hodge numbers) were given in [37]. This is the motivation and the starting point for the work developed in this thesis.

E-polynomials

When X is a compact Kähler manifold, a classical result concerning the cohomology of X is the *Hodge decomposition*; it asserts that there is a direct sum decomposition

$$H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}$$

such that $\overline{H^{p,q}} = H^{q,p}$. These complex subspaces $H^{p,q}$ have a sheaf theoretic interpretation in terms of Dolbeault cohomology and can be characterized as the space of de Rham cohomology classes that can be represented by complex (p,q)-forms. A generalization was given by Deligne [17, 18] for algebraic quasiprojective varieties, such as our character varieties. In that case, for each k the cohomology of X with complex coefficients, $H^k(X,\mathbb{C})$, is endowed with two filtrations, an ascending *weight* filtration

$$0 = W_{-1} \subseteq W_0 \subseteq \ldots \subseteq W_{2k} = H^{2k}(X,\mathbb{C})$$

and a descending *Hodge* filtration

$$H^k(X,\mathbb{C}) = F^0 \supseteq F^1 \supseteq \ldots \supseteq F^{n+1} = 0$$

that allow to define *mixed Hodge numbers* $h^{k,p,q}(X)$ from the graded pieces,

$$h^{k,p,q}(X) = \dim_{\mathbb{C}}(Gr^p_F Gr^W_{p+q} H^k(X,\mathbb{C})).$$

This structure is what is called a *mixed Hodge structure*, and it is called *pure* when the weight filtration is trivial, $0 = W_{-1} \subseteq W_k = H^k(X,\mathbb{C})$, as it happens in the compact Kähler case.

If we look at the cohomology of the moduli spaces that appear in the non-abelian Hodge correspondence, the mixed Hodge structures coincide for $\mathcal{M}_{Dol}$ and $\mathcal{M}_{dR}$ and moreover, they are pure [34] due to the existence of smooth compactifications for both spaces. However, the mixed Hodge structure on $H^*(\mathcal{M}_B)$ is not pure, so the Betti moduli space cannot be isomorphic as a complex algebraic variety to the previous spaces, even though the Riemann-Hilbert map taking a flat connection to its holonomy provides an isomorphism of complex analytic manifolds. Therefore, there are two weight filtrations on $H^k(\mathcal{M}_{Dol},\mathbb{C})$: one arising from the pure mixed Hodge structure of $\mathcal{M}_{Dol}$ and another coming from the mixed Hodge structure of $\mathcal{M}_B$ via the non-abelian Hodge correspondence. In [15], de Cataldo, Hausel and Migliorini gave an interpretation of this fact by proving that the second filtration coincides with the perverse Leray filtration associated to the Hitchin fibration $\mathcal{M}_{Dol} \to \mathcal{A}$ in the rank 2 case.

Motivated by mirror symmetry and by the previous considerations, E-polynomials of these spaces were taken into consideration. From the compactly supported mixed Hodge numbers $h_c^{k,p,q}$ of X, the E-polynomial is defined taking Euler characteristics,

$$\chi_c^{p,q}(X) := \sum_k (-1)^k h_c^{k,p,q}(X)$$

and forming the polynomial

$$e(u,v)(X) := \sum_{p,q} \chi_c^{p,q}(X) u^p v^q$$

Observe that when the mixed Hodge structure is pure, we recover all mixed Hodge numbers from the E-polynomial. In [38], these polynomials were computed for the Dolbeault moduli space for $SL(n,\mathbb{C})$ and $PGL(n,\mathbb{C})$ for $n=2,3$ and it was conjectured that they also agreed for arbitrary $n \in \mathbb{N}$. To be more precise, the original statement uses stringy E-polynomials, a twisted version that is required to take into account the singularities of the spaces, a version that in any case coincides with the ordinary E-polynomials in the smooth case.

E-polynomials of character varieties were discussed by Hausel and Rodriguez-Villegas in [37]. They were able to compute the E-polynomials of the twisted Betti moduli space $\mathcal{M}_B^d$ for $GL(n,\mathbb{C})$ using powerful arithmetic techniques. The first ingredient of their approach is a theorem by Katz that relates the E-polynomial with the number of points of the variety over finite fields. If we consider any smooth quasi-projective variety X defined over $\mathbb{Z}$, we say that X is of *polynomial count* if the number of points of X over a finite field $\mathbb{F}_q$ is given by a polynomial $P(q)$. Katz's theorem states that for varieties of polynomial count, we can obtain their E-polynomials from P, namely

$$e(u,v) = P(uv) := \#\{X(\mathbb{F}_q)\}$$

where $q = uv$. The result becomes specially useful for varieties where the mixed Hodge structure is of Hodge-Tate type, such as our character varieties (when $h^{k,p,q} = 0$ for $p+q \neq k$, they are sometimes named of balanced type). The second ingredient is to compute the number of points of these varieties over finite fields, using arithmetic tools and the character formula

$$P(q) = \#(\mathcal{M}_B^d(\mathbb{F}_q)) = \sum_{\chi \in Irr(GL(n,\mathbb{F}_q))} \frac{|GL(n,\mathbb{F}_q)|^{2g-2}}{\chi(I_n)^{2g-1}} \chi(\xi_n I_n)$$

where χ ranges across all irreducible characters. The final calculation involves character tables of $GL(n,\mathbb{F}_q)$ and provides a generating function for the E-polynomials. For $n=2$, using the explicit description of the cohomology ring of $\mathcal{M}_{Dol}$ given in [40, 39], they were able to determine their mixed Hodge polynomials. Many other interesting conjectures were derived from this work, such as general formulas for the mixed Hodge polynomials of these spaces and certain cohomological relations, such as a curious Poincar duality or a curious Hard Lefschetz theorem. It is also worth mentioning the Purity Conjecture, that relates the pure part of the mixed Hodge structure of the character variety with the A-polynomial of certain quivers.

More results in this direction were later obtained by Hausel, Letellier and Rodriguez-Villegas in a series of papers [35, 36], for $GL(n,\mathbb{C})$-character varieties associated to punctured Riemann surfaces with generic semisimple elements at the punctures. Again, their E-polynomials were given in terms of a generating function involving McDonald polynomials. Further connections with the representation theory of quivers and Hilbert schemes were introduced. For $SL(n,\mathbb{C})$, Mereb obtained an analogous generating function for the E-polynomials of the twisted character variety [60].

Geometric method

An alternative method for the computation of the E-polynomials of character varieties was given in [53] by Logares, Muñoz and Newstead. In this case, the approach was geometric and it was based on the study of the behaviour of E-polynomials for fibrations which are locally trivial in the analytic topology but not in the Zariski topology. It also uses several convenient stratifications of the moduli space due to the fact that E-polynomials are additive with respect to them. The main results in [53] are explicit formulas for the E-polynomials of $SL(2,\mathbb{C})$-character varieties of low genus surfaces ($g = 1, 2$) with one puncture. In this case, this technique allows to deal with the cases where the monodromy around the puncture is not diagonalizable (and there is no correspondence with a Higgs moduli space). When the monodromy is given by semisimple elements, the corresponding Betti moduli space is known as *parabolic*, and it is homeomorphic to the moduli space of Higgs bundles with parabolic structures at the punctures [75]. The behaviour of these parabolic moduli spaces for genus 1, $\mathcal{M}_\lambda^{g=1}(\lambda \neq 0, \pm 1)$ as λ varies is encoded in what is called the *Hodge monodromy representation*, a suitable way to treat the information given by the E-polynomials of the fibre and the total space of the fibration given by the family $\mathcal{M}_\lambda \to \mathbb{C} - \{0, \pm 1\}$. This geometric point of view is the one considered and developed in this thesis.

Objectives

The main objective of the present dissertation is to extend the geometric study of the E-polynomials of character varieties of complex curves of low genus described in [53]. The first goal is to apply the techniques described therein to other $SL(2,\mathbb{C})$-character varieties, in order to find geometric descriptions of these spaces that allow to compute their E-polynomials and to obtain new topological information. The second main objective is to study the $SL(2,\mathbb{C})$-character variety associated to a complex curve of genus 3. It is soon noted that the tools described in [53] need to be extended for fibrations where the base is of dimension higher than one, so another goal is to provide the necessary framework to solve this problem, since the complexity of these spaces grow as soon as the genus increases. Besides, the existing formulas in the literature cover do not cover all the cases and rely on aritmetic methods, based on counting points over finite fields, so any relation of the E-polynomials with the geometry of these spaces is desired. The third and final goal is to obtain closed and explicit formulas for arbitrary genus and arbitrary monodromy, in order to shed some light on the problem and confirm certain phenomena that were already observed for low genus.

Results

Chapter 1 introduces basic definitions and character varieties focusing on a particular problem as a guiding example, $SU(2)$-character varieties of torus knot groups. These groups admit the following presentation

$$G_{m,n} = \langle x, y \mid x^m = y^n \rangle,$$

where $m, n \in \mathbb{N}$ are coprime. These character varieties were treated in [64] and [66] using combinatorial tools for the case $(m, 2)$ and $G = SL(2,\mathbb{C}), SU(2)$. For arbitrary (m, n) and $G = SL(2,\mathbb{C})$, they were geometrically described in [61] using characters. Character varieties for other knot groups have been studied by several authors, as a method to obtain knot invariants [8, 42, 43, 44, 45].

The locus of irreducible representations in $\mathcal{M}_{G_{m,n}}(SL(2,\mathbb{C}))$ is a collection of $\frac{(m-1)(n-1)}{2}$ complex lines whose closure intersects the reducible part, isomorphic to $\mathbb{C}$. Here, we analyze the injection

$$i_* : \mathcal{M}_{G_{m,n}}(SU(2)) \longrightarrow \mathcal{M}_{G_{m,n}}(SL(2,\mathbb{C}))$$

The set of reducible representations in $\mathcal{M}_{G_{m,n}}(SU(2))$ is isomorphic to a real closed interval, whereas the irreducible locus consists of $\frac{(m-1)(n-1)}{2}$ open real intervals sitting inside each copy of $\mathbb{C}$ in $\mathcal{M}^{irr}_{G_{m,n}}(SL(2,\mathbb{C}))$. A similar description holds for arbitrary $m, n \in \mathbb{N}$. From the geometric description, we obtain as a corollary

Corollary 1.3.5. *$\mathcal{M}_{G_{m,n}}(SU(2))$ is a deformation retract of $\mathcal{M}_{G_{m,n}}(SL(2,\mathbb{C}))$.*

At the moment, it is known that given G a complex Lie group and $K \subset G$ a maximal compact subgroup, there are certain groups for which the K-character variety is a deformation retract of the G-character variety [24, 26], and that are others for which it is not [5]. The characterization of those groups for which the question is affirmative is still an open question.

The second chapter of this dissertation introduces Hodge theory of algebraic varieties and defines and states all basic properties of E-polynomials. Moreover, the main tool to handle fibrations $F \to E \to B$ that are not Zariski locally trivial and such that the mixed Hodge structure is of Hodge-Tate type is given. When the monodromy is finite and abelian, i.e. if it factors through such a group Γ, each $H^{k,p,p}(F)$ can be regarded as a module over the representation ring $R(\Gamma)$. The Hodge monodromy representation of the fibration is a polynomial with coefficients in the representation ring,

$$R(E) := \sum (-1)^k H_c^{k,p,p}(F) q^p \in R(\Gamma)[q],$$

that keeps track of the cohomological information given by the fibre, the total space of the fibration and the monodromy. Chapter 2 extends the results in [53] for fibrations where the base is one dimensional (Corollary 2.3.5) to higher dimensional bases (Theorem 2.3.2, Corollary 2.3.7), which is later used in Chapter 4 to compute the E-polynomials of character varieties of surfaces of genus 3. Besides, Theorem 2.3.2 may be of interest to tackle the problem for higher rank.

E-polynomials of character varieties of complex curves of genus 1, as well as their Hodge monodromy representations, were computed in [53]. They are repeatedly used throughout this dissertation and since they are needed to solve the problem for arbitrary genus, we call them building blocks. To obtain their E-polynomials, one needs to take explicit equations and obtain slices for the conjugacy action, as well as to stratify and study the monodromy of each stratum. In Chapter 3, we illustrate this technique with a similar problem, character varieties associated to non-orientable surfaces of low genus. They have been recently treated from the point of view of Higgs bundles, and it has been shown that a non-abelian Hodge correspondence holds [49]. Their E-polynomials are computed and some topological information of these spaces is obtained as a consequence. The result is condensed in the following theorems.

Theorem 3.1.1. *Let K be the Klein bottle. The E-polynomials of the $SL(2,\mathbb{C})$-character varieties $\mathcal{M}_C(K)$ are*

$$e(\mathcal{M}_{\mathrm{Id}})(K) = 3q - 2,$$
$$e(\mathcal{M}_{-\mathrm{Id}})(K) = q - 1,$$

$$
\begin{aligned}
e(\mathcal{M}_{J_+})(K) &= q^2 + 2q - 7, \\
e(\mathcal{M}_{J_-})(K) &= q^2 + 3q, \\
e(\mathcal{M}_{\xi_\lambda})(K) &= q^2 + 2q + 1.
\end{aligned}
$$

Theorem 3.1.2. *Let Σ be the connected sum of three projective planes. The E-polynomials of its associated $SL(2,\mathbb{C})$-character varieties are*

$$
\begin{aligned}
e(\mathcal{M}_{\mathrm{Id}})(\Sigma) &= q^3 - 6q - 1, \\
e(\mathcal{M}_{-\mathrm{Id}})(\Sigma) &= 2q^3 + 7q^2 - 1, \\
e(\mathcal{M}_{J_+})(\Sigma) &= q^5 + 5q^3 + 12q^2 - 8q + 26, \\
e(\mathcal{M}_{J_-})(\Sigma) &= q^5 - 5q^3 - 12q^2, \\
e(\mathcal{M}_{\xi_\lambda})(\Sigma) &= q^5 + q^4 + 2q^3 + 8q^2 - 27q + 23.
\end{aligned}
$$

Chapters 4 and 5 can be considered the core of this phD dissertation and deal with $SL(2,\mathbb{C})$-character varieties of complex curves of genus $g \geq 3$. In Chapter 4, the E-polynomials of the ordinary character variety $\mathcal{M}_{\mathrm{Id}}$ and the twisted character variety $\mathcal{M}_{-\mathrm{Id}}$ are computed for $g = 3$. In order to do so, the space of representations in $SL(2,\mathbb{C})^6$ is conveniently stratified and it is at this point where fibrations over two dimensional bases appear. The specific case $g = 3$ serves as the starting point of the induction for arbitrary genus that is carried out in Chapter 5. The basic idea is to decompose X^g as a connected sum $X^g = X^{g-1} \# X^1$. From X^g, one obtains information for X^{g-1} with a hole, which is used to compute the E-polynomial corresponding to $X^{g+1} = X^{g-1} \# X^2$. The building blocks for $g = 1, 2$ and the genus 2 Hodge monodromy representation computed in Chapter 4 come into play at this point.

The main results are

Theorem 5.1.1. *Let X be a complex curve of genus $g \geq 1$. Let $\mathcal{M}^g_C = \mathcal{M}^g_C(SL(2,\mathbb{C}))$ be the character variety corresponding to $C \in SL(2,\mathbb{C})$. The E-polynomials of $\mathcal{M}^g_C$ are:*

$$
\begin{aligned}
e(\mathcal{M}^g_{\mathrm{Id}}) &= (q^3 - q)^{2g-2} + (q^2 - 1)^{2g-2} - q(q^2 - q)^{2g-2} - 2^{2g} q^{2g-2} \\
&\quad + \frac{1}{2} q^{2g-2} (q + 2^{2g} - 1)((q+1)^{2g-2} + (q-1)^{2g-2}) \\
&\quad + \frac{1}{2} q((q+1)^{2g-1} + (q-1)^{2g-1}). \\
e(\mathcal{M}^g_{-\mathrm{Id}}) &= (q^3 - q)^{2g-2} + (q^2 - 1)^{2g-2} - 2^{2g-1}(q^2 + q)^{2g-2} + (2^{2g-1} - 1)(q^2 - q)^{2g-2}. \\
e(\mathcal{M}^g_{J_+}) &= (q^3 - q)^{2g-2}(q^2 - 1) + (2^{2g-1} - 1)(q - 1)(q^2 - q)^{2g-2}
\end{aligned}
$$

$$
\begin{aligned}
&- 2^{2g-1}(q+1)(q^2+q)^{2g-2} + \frac{1}{2}q^{2g-2}(q-1)\left((q-1)^{2g-1} - (q+1)^{2g-1}\right).\\
e(\mathcal{M}^g_{J_-}) &= (q^3-q)^{2g-2}(q^2-1) + (2^{2g-1}-1)(q-1)(q^2-q)^{2g-2}\\
&+ 2^{2g-1}(q+1)(q^2+q)^{2g-2}\\
e(\mathcal{M}^g_{\xi_\lambda}) &= (q^3-q)^{2g-2}(q^2+q) + (q^2-1)^{2g-2}(q+1) + (2^{2g}-2)(q^2-q)^{2g-2}q,
\end{aligned}
$$

for $J_+ = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $J_- = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$ *and* $\xi_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, $\lambda \neq 0, \pm 1$, *where* $q = uv$.

Theorem 5.1.2. *All character varieties* $\mathcal{M}_C(SL(2,\mathbb{C}))$ *are of balanced type.*

The behaviour of the parabolic character variety for arbitrary genus is given by its Hodge monodromy representation,

Theorem 5.1.4. *Let* X *be a curve of genus* $g \geq 1$. *Then*

$$
\begin{aligned}
R(\mathcal{M}^g_{\xi_\lambda}) = &\left((q^3-q)^{2g-2}(q^2+q) + (q+1)(q^2-1)^{2g-2} - q(q^2-q)^{2g-2}\right) T\\
&+ \left((2^{2g}-1)q(q^2-q)^{2g-2}\right) N,
\end{aligned}
$$

where the E-polynomial of the invariant part of the cohomology is the polynomial accompanying T, *and the E-polynomial of the non-invariant part is the polynomial accompanying* N, *where* T, N *are the trivial and non trivial representations respectively.*

Finally, we derive some implications from Theorem 5.1.1.

Corollary 5.9.1. *Let* X *be a complex curve of genus* $g \geq 2$. *The Euler characteristic of* $\mathcal{M}^g_C = \mathcal{M}^g_C(SL(2,\mathbb{C}))$ *is given by*

$$
\begin{aligned}
\chi(\mathcal{M}^g_{\mathrm{Id}}) &= 2^{4g-3} - 3 \cdot 2^{2g-2},\\
\chi(\mathcal{M}^g_{-\mathrm{Id}}) &= -2^{4g-3},\\
\chi(\mathcal{M}^g_{J_+}) &= -2^{4g-2},\\
\chi(\mathcal{M}^g_{J_-}) &= 2^{4g-2},\\
\chi(\mathcal{M}^g_{\xi_\lambda}) &= 0.
\end{aligned}
$$

Corollary 5.9.2. *Let* X *be a complex curve of genus* $g \geq 2$. *Then* $\mathcal{M}^g_{\mathrm{Id}}$ *and* $\mathcal{M}^g_{-\mathrm{Id}}$ *are of dimension* $6g-6$ *and* $\mathcal{M}^g_{J_+}$, $\mathcal{M}_{J_-}$ *and* $\mathcal{M}^g_{\xi_\lambda}$ *are of dimension* $6g-4$. *All of them have a unique component of maximal dimension.*

Corollary 5.9.3. *Let* X *be a complex curve of genus* $g \geq 1$. *Then* $e(\mathcal{M}^g_{-\mathrm{Id}})$, $e(\mathcal{M}^g_{\xi_\lambda})$, *and its invariant and non-invariant part given in Theorem 4.3 are palindromic polynomials.*

Conclusions

The geometric study of the E-polynomials of $SL(2,\mathbb{C})$-character varieties associated to complex curves of arbitrary genus has been successfully developed in this thesis. Theorems 5.1.1, 5.1.2 and 5.1.4 are the main results of this dissertation and accomplish the objectives that were set at the beginning. The first objective is fulfilled studying the case of character varieties associated to non-orientable surfaces of low genus, where the techniques that were stablished in [53] apply. Another interesting problem regarding a family of $SL(2,\mathbb{C})$-character varieties, associated to torus knots, is studied in Chapter 1: the geometric description in terms of characters provides an affirmative answer to the question of when the $SU(2)$-character variety is a retraction of the corresponding $SL(2,\mathbb{C})$-character variety. Although the general problem remains open, it provides a new example which is neither abelian nor free.

The genus 3 case is solved studying fibrations over 2-dimensional bases and this suffices for arbitrary genus, although Theorem 2.3.2 applies to any fibration where the monodromy is abelian and finite, without any dimensional restrictions. It might be useful for many other situations. The final and main objective is accomplished in Chapter 5, where formulas for arbitrary monodromy and arbitrary genus are given. They generalize the formulas given in [53] for $g = 1, 2$. The induction process also provides a remarkable novelty: the "E-polynomial" information of the character variety of genus g is encoded in the eight polynomials $(e_0^g, e_1^g, e_2^g, e_3^g, a_g, b_g, c_g, d_g)$ (the first four encode the information corresponding to monodromies Id, $-\mathrm{Id}$, J_+ and J_-, and the last four correspond to the information of the parabolic case, which is given in the Hodge monodromy representation) and the information for genus $g+1$ is obtained in terms of a linear map, given by a certain matrix of E-polynomials. In other words, the topological procedure of attaching a handle to X_g in order to obtain X_{g+1} is reproduced by this linear map at the level of E-polynomials.

The ideas and tools used to obtain the results in Chapter 5 can be applied to other settings. As a first step, it certainly allows to attack the study of $PGL(2,\mathbb{C})$-character varieties of complex curves of arbitrary genus and also to extend to arbitrary genus the results obtained in Chapter 3 for non-orientable surfaces of genus 1 and 2. These two extensions will be presented in future work. Moreover, the techniques developed in Chapter 2 may be applied in the future to study other groups, such as $G = SL(3,\mathbb{C})$. Although some technical difficulties appear in the computation of the basic pieces of low genus, the induction procedure could be carried again to tackle the general genus case. This would be a definite breakthrough for the arbitrary rank case.

The main results of this thesis are collected on the preprints [56, 57, 58]. The first one contains the results about $SU(2)$-character varieties of torus knots that are given in Chapter 1, whereas the second and the third present the case of $SL(2,\mathbb{C})$-character varieties of curves of genus 3 and arbitrary genus respectively.

Chapter 1

Character varieties. The torus knot groups case

1.1 Preliminaries and notation

Given a finitely presented group $\Gamma = \langle x_1 \ldots x_l | r_1, ..., r_s \rangle$ and a reductive algebraic group G, a G-representation is a homomorphism $\rho : \Gamma \to G$. Every representation is completely determined by the image of the generators: if we write $A_j = \rho(x_j)$, it is determined by the k-tuple $(A_1, ..., A_k) \in G^k$ satisfying the relations $r_j(A_1, ..., A_k) = \mathrm{Id}$. Since G is algebraic, it follows from the definitions that the space of all representations,

$$R_G(\Gamma) := \mathrm{Hom}(\Gamma, G)$$

is an affine algebraic set (complex or real depending on the nature of G).

It is natural to declare a certain equivalence relation between these representations: we say that ρ and ρ' are equivalent if there exists $P \in G$ such that $\rho'(g) = P^{-1}\rho(g)P$ for all $g \in \Gamma$.

Definition 1.1.1. *We define the G-character variety of Γ to be the GIT quotient*

$$\mathcal{M}_G(\Gamma) = \mathrm{Hom}\,(\Gamma, G)//G.$$

By a GIT quotient, we refer to the scheme corresponding to the spectrum of the ring of invariant functions. If $R_G(\Gamma) = Spec\, R$, then $\mathcal{M}_G = Spec\, R^G$.

Remark 1.1.2. *Note that different conjugacy classes may correspond to the same point when the GIT quotient is made. For example, taking $G = \mathbb{Z}$, the representation defined by*

$$\rho(1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

is in the same equivalence class as the trivial representation, although they do not belong to the same conjugacy class. This happens because we identify orbits whose closures intersect. Conjugating ρ *by* $H = \left(\begin{smallmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{smallmatrix}\right)$ *we get that*

$$H\rho H^{-1}(1) = \begin{pmatrix} 1 & \frac{1}{\mu^2} \\ 0 & 1 \end{pmatrix},$$

whose limit, when $\mu \to \infty$*, is the trivial representation.*

Definition 1.1.3. *A representation* $\rho : \Gamma \to GL(n, \mathbb{C})$ *is*

- *Irreducible, if the only* Γ*-invariant subspaces of* $\mathbb{C}^n$ *are* $\{0\}$ *and itself,* $\mathbb{C}^n$*.*
- *Completely reducible, if* $\mathbb{C}^n$ *decomposes into a sum of irreducible* Γ*-modules. This means that in a certain basis the representation lies in the subset of block diagonal matrices in* $GL(n, \mathbb{C})$*.*

These definitions can be generalized for any representation into a general algebraic group G.

Definition 1.1.4. *We say that a subgroup* $H \leq G$ *is*

- *Irreducible, when* H *is not contained in any parabolic subgroup* P*.*
- *Completely reducible, when for any parabolic* P *such that* $H \subseteq P$*, there is a Levi subgroup* L *such that* $H \subseteq L \subseteq P$ *(recall that a Levi subgroup* L *of an algebraic group* G *is a connected subgroup such that* G *is the semi-direct product of* L *and the unipotent radical of* G*).*

In this context, we say that a representation $\rho : \Gamma \longrightarrow G$ *is irreducible (completely reducible) when* $\rho(\Gamma) \subseteq G$ *is.*

Both notions coincide for the groups that we will treating in this chapter and subsequent ones, so we will make no distinction between them.

We focus in this chapter on the case when Γ is a torus knot group and $G = SL(2, \mathbb{C})$ or $G = SU(2)$. Consider the torus of revolution $T^2 \subset S^3$. We identify it with $\mathbb{R}^2/\mathbb{Z}^2$, where $\mathbb{Z} = \langle (1,0), (0,1) \rangle$, via the map

$$\begin{aligned} F : \mathbb{R}^2/\mathbb{Z}^2 &\longrightarrow T^2 \subset \mathbb{R}^3 \subset S^3 \\ (x, y) &\longrightarrow ((2 + \cos 2\pi x)\cos 2\pi y, (2 + \cos 2\pi x)\sin 2\pi y, \sin 2\pi x) \end{aligned}$$

The image of the line $y = \frac{m}{n}x$ defines the torus knot of type (m, n), $K_{m,n} \subset S^3$ for coprime m, n. An important invariant of a knot is the fundamental group of its complement in S^3, here $G_{m,n} = \pi_1(S^3 - K_{m,n})$. These groups admit the following presentation

$$G_{m,n} = \langle x, y \mid x^m = y^n \rangle. \tag{1.1}$$

The $SL(2, \mathbb{C})$-character variety of these groups for the case $(m, 2)$ was treated in [65]. A complete description for (m, n) coprime was given in [61], and the general case (m, n) was studied using combinatorial tools in [54]. $SU(2)$-character varieties for knot groups were studied in [51]. For the case $(m, 2)$, the relation between both character varieties has been recently treated in [66].

1.2 $SU(2)$ and $SL(2, \mathbb{C})$-character varieties of torus knots

Let Γ be any finitely presented group. We can start looking at $SL(2, \mathbb{C})$-representations of Γ, which form the representation space $R_{SL(2,\mathbb{C})}(\Gamma)$, and construct the associated moduli space

$$\mathcal{M}_{SL(2,\mathbb{C})}(\Gamma) = \mathrm{Hom}(\Gamma, SL(2, \mathbb{C}))//SL(2, \mathbb{C})$$

where we identify representations that are $SL(2, \mathbb{C})$-equivalent. On the other hand, the natural inclusion $SU(2) \hookrightarrow SL(2, \mathbb{C})$ shows that we can regard every $SU(2)$- representation as a $SL(2, \mathbb{C})$-representation. Moreover, if two representations are $SU(2)$-equivalent, then they are also $SL(2, \mathbb{C})$-equivalent. This leads to a map between moduli spaces:

$$\mathcal{M}_{SU(2)}(\Gamma) \xrightarrow{i_*} \mathcal{M}_{SL(2,\mathbb{C})}(\Gamma)$$

Definition 1.2.1. *Given a representation $\rho \in R_{SL(2,\mathbb{C})}(\Gamma)$, we define its character χ_ρ as the map*

$$\begin{aligned} \chi_\rho : \Gamma &\to \mathbb{C} \\ g &\longrightarrow \mathrm{tr}(\rho(g)) \end{aligned}$$

The map $\chi : R_{SL(2,\mathbb{C})}(\Gamma) \to \mathbb{C}^\Gamma$ taking its representation to its character is called the *character map.* Note that equivalent representations have the same character.

Definition 1.2.2. *The image of the character map,*

$$X_{SL(2,\mathbb{C})}(\Gamma) = \chi(R_{SL(2,\mathbb{C})}(\Gamma))$$

is also called the character variety of Γ in the literature.

For the torus knot groups case, it is seen in [14] that:

- $X_{SL(2,\mathbb{C})}(\Gamma)$ can be endowed with the structure of algebraic variety.
- The natural map that takes every representation to its character, $\mathcal{M}_{SL(2,\mathbb{C})}(\Gamma) \longrightarrow X_{SL(2,\mathbb{C})}(\Gamma)$, is bijective. In fact, when Γ is a torus knot group, it is an isomorphism of algebraic varieties, and it is shown in [61] directly as a consequence of the geometric description of the character variety.

For other algebraic groups G and different Γ, the correspondence is not always bijective. The ring of invariant functions of the character variety, $\mathbb{C}[\mathcal{M}_G(\Gamma)]$, is generated by traces in some cases:

- $G = SL(2,\mathbb{K})$, $\mathbb{C}[X_G(\Gamma)] = \langle \chi_{x_i}, \chi_{x_i x_j}, \chi_{x_i x_j x_k}, i,j,k = 1 \ldots r\rangle$ [7, 69], see also [30]. Sometimes fewer generators may even suffice, for example if Γ is abelian, then $\mathbb{C}[X_G(\Gamma)] = \langle \chi_{x_i}, i = 1 \ldots r\rangle$.
- $G = PSL(2,\mathbb{C})$, see [41].
- $G = SL(n,\mathbb{K})$, $\mathbb{C}[X_G(\Gamma)] = \langle \chi_{\gamma_i} \mid \gamma_i \in \mathcal{B}'\rangle$, where B' is the set of words with basis the generators x_i of a certain bounded lenght (the upper bound given by a term that depends on Γ) [72].
- Classical groups such as $G = SO(2n+1,\mathbb{K}), Sp(n,\mathbb{K}), O(n,\mathbb{K})$ [25, 72].

However, there are examples where the subalgebra generated by characters is strictly smaller than the ring of invariant functions, as $G = SO(2n,\mathbb{C})$, see [73].

The goal of this chapter is to analyze what happens for $G = SU(2)$ and the relationship between the $SL(2,\mathbb{C})$ and the $SU(2)$-character varieties. We emphasize that $X_{SL(2,\mathbb{C})}(\Gamma)$, as a set, consists of characters of $SL(2,\mathbb{C})$-representations. We can also take the set of characters of $SU(2)$-representations, and again we will have a map

$$X_{SU(2)}(\Gamma) \xrightarrow{i^*} X_{SL(2,\mathbb{C})}(\Gamma).$$

We recall that $SU(2) \cong S^3$, the isomorphism being given by

$$\begin{array}{rcl} S^3 \subset \mathbb{C}^2 & \longrightarrow & SU(2) \\ (a,b) & \longrightarrow & \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \end{array}$$

The correspondence is a ring homomorphism if we look at S^3 as the set of unit quaternions. First of all, we want to point out the following fact, which was already true for $SL(2,\mathbb{C})$:

Proposition 1.2.3. *The correspondence*

$$\begin{array}{ccc} \mathcal{M}_{SU(2)}(\Gamma) & \longrightarrow & X_{SU(2)}(\Gamma) \\ \rho & \longrightarrow & \chi_\rho \end{array}$$

that takes a representation to its character is bijective.

Proof. We follow the steps taken in [14], this time for $SU(2)$. First of all, every matrix A in $SU(2)$ is normal, hence diagonalizable. Since $\det(A) = 1$, the eigenvalues of A are $\{\lambda, \lambda^{-1}\}$ for some $\lambda \in \mathbb{C}^*$. In particular, $\operatorname{tr}(A)$ completely determines the set of eigenvalues $\{\lambda, \lambda^{-1}\}$.

Now, if ρ is a reducible $SU(2)$-representation, there is a common eigenvector e_1 for all $\rho(g)$ and therefore they are all diagonal with respect to the same basis. If ρ' is a second reducible representation such that $\chi_\rho(g) = \chi_{\rho'}(g)$ for all $g \in G$, this means that they share the same eigenvalues for every $g \in G$. After choosing another basis for ρ' such that $\rho'(g)$ is diagonal for all $g \in G$,

$$\rho(g) = \begin{pmatrix} \lambda(g) & 0 \\ 0 & \lambda^{-1}(g) \end{pmatrix}, \quad \rho'(g) = \begin{pmatrix} \mu(g) & 0 \\ 0 & \mu^{-1}(g) \end{pmatrix},$$

where either $\lambda(g) = \mu(g)$ or $\lambda(g) = \mu^{-1}(g)$ for every $g \in G$. Interchanging the roles of λ and λ^{-1} if necessary, there is always $g_1 \in G$ such that $\lambda(g_1) = \mu(g_1)$, so there is $g_1 \in G$ such that $\rho(g_1) = \rho'(g_1)$. We also notice that if $\rho(g) = \pm\operatorname{Id}$, then $\rho'(g) = \rho(g) = \pm\operatorname{Id}$.

We claim that $\rho(g_2) = \rho'(g_2)$ for all $g_2 \in G$. If not, there exists $g_2 \in G$ such that $\rho(g_2) = \rho'(g_2)^{-1} \neq \pm\operatorname{Id}$. So $\lambda(g_1) = \mu(g_1)$ and $\lambda(g_2) = \mu^{-1}(g_2)$. On the other hand, we know that $\operatorname{tr}(\rho'(g_1g_2)) = \operatorname{tr}(\rho(g_1g_2))$, so

$$\begin{aligned} \mu(g_1)\mu(g_2) + \mu^{-1}(g_1)\mu^{-1}(g_2) &= \lambda(g_1)\lambda(g_2) + \lambda^{-1}(g_1)\lambda^{-1}(g_2) \\ &= \mu(g_1)\mu^{-1}(g_2) + \mu^{-1}(g_1)\mu(g_2). \end{aligned}$$

Rearranging the terms

$$\mu(g_2)(\mu(g_1) - \mu^{-1}(g_1)) = \mu^{-1}(g_2)(\mu(g_1) - \mu^{-1}(g_1)),$$

which implies that $\mu(g_2) = \pm 1$, so that $\rho(g_2) = \pm\operatorname{Id}$, a contradiction. Therefore $\lambda(g) = \mu(g)$ for all $g \in G$. Hence there exists $P \in SU(2)$ such that $\rho(g) = P^{-1}\rho(g)P$ for all $g \in G$, i.e, the representations are equivalent.

For the irreducible case, we point out the following fact: if ρ is a irreducible $SU(2)$-representation and $\rho(g) \neq \pm\operatorname{Id}$ for a given $g \in G$, then there exists $h \in G$ such that ρ restricted to the subgroup $H = \langle g, h\rangle$ is again irreducible. To see it, since $\rho(g) \neq \pm\operatorname{Id}$, $\rho(g)$ has two eigenspaces L_1, L_2 associated to the pair of different eigenvalues μ_1, μ_2. Since the representation is irreducible, there are elements h_i such that L_i is not invariant under $\rho(h_i)$.

We can take $h = h_1$ or $h = h_2$ unless L_1 is invariant under $\rho(h_2)$, or L_2 is invariant under $\rho(h_1)$, in this case we can choose $h = h_1 h_2$.

For a group generated by two elements, $H = \langle g, h\rangle$, the reducibility of a representation is completely determined by $\chi_\rho([g,h])$. It can be seen in the following chain of equivalences,

$$\begin{aligned} \rho|_H \text{ is reducible} \quad &\Leftrightarrow \quad \rho(g), \rho(h) \text{ share a common eigenvector} \\ &\Leftrightarrow \quad \rho(g), \rho(h) \text{ are simultaneously diagonalizable} \\ &\Leftrightarrow \quad [\rho(g), \rho(h)] = \mathrm{Id} \\ &\Leftrightarrow \quad \mathrm{tr}[\rho(g), \rho(h)] = 2 \\ &\Leftrightarrow \quad \chi_\rho([g,h]) = 2. \end{aligned}$$

Let ρ, ρ' be two $SU(2)$-representations such that $\chi_\rho = \chi_{\rho'}$. By the previous observation, there are $g, h \in G$ such that $\rho|_{\langle g,h\rangle}$ is irreducible, i.e, $\chi_\rho([g,h]) \neq 2$. It follows that, since $\chi_\rho = \chi_{\rho'}$, $\chi_{\rho'}([g,h]) \neq 2$, so $\rho'|_{\langle g,h\rangle}$ is irreducible too. Varying ρ, ρ' in their equivalence classes, we can assume that there are basis B, B' such that

$$\rho(h) = \rho'(h) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}.$$

The matrices $\rho(g), \rho'(g)$ will not be diagonal, by irreducibility, and conjugating again by diagonal unitary matrices, we can assume that

$$\rho(g) = \begin{pmatrix} a & -b \\ b & \bar{a} \end{pmatrix}, \qquad \rho'(g) = \begin{pmatrix} a' & -b' \\ b' & \bar{a}' \end{pmatrix},$$

for $a, a' \in \mathbb{C},\ b, b' \in \mathbb{R}^+$. Notice that $b, b' \neq 0$ since $\rho|_{\langle g,h\rangle}$ is irreducible. In general, for any $\alpha \in G$

$$\rho(\alpha) = \begin{pmatrix} x & -\bar{y} \\ y & \bar{x} \end{pmatrix}, \qquad \rho'(\alpha) = \begin{pmatrix} x' & -\bar{y}' \\ y' & \bar{x}' \end{pmatrix}.$$

Now, the equations $\chi_\rho(\alpha) = \chi_{\rho'}(\alpha)$, $\chi_\rho(h\alpha) = \chi_{\rho'}(h\alpha)$ imply that:

$$\begin{aligned} x + \bar{x} &= x' + \bar{x}' \\ \lambda x + \lambda^{-1}\bar{x} &= \lambda x' + \lambda^{-1}\bar{x}' \end{aligned}$$

and since $\lambda \neq \pm 1$, we get that $x = x'$.

Substituting $\alpha = g$, we get that $a = a'$ and since $\det(\rho(g)) = \det(\rho'(g)) = 1$, $b = b'$, so $\rho(g) = \rho'(g)$.

Substituting again $g\alpha$ for α, we arrive at the equation $ax - by = ax - by'$, which implies that $y = y'$ and finally that $\rho(\alpha) = \rho'(\alpha)$: we have proved that the representations ρ and ρ', after $SU(2)$-conjugation, are equivalent. $\square$

Remark 1.2.4. *As a consequence of Proposition 1, the moduli space is precisely the set of conjugacy classes of representations, i.e., there are no extra identifications as in the $SL(2,\mathbb{C})$-case.*

Corollary 1.2.5. *We have a commutative diagram*

$$\begin{tikzcd} \mathcal{M}_{SU(2)}(\Gamma) \arrow[r, "1:1"] \arrow[d, "i_*"] & X_{SU(2)}(\Gamma) \arrow[d, "i_*"] \\ \mathcal{M}_{SL(2,\mathbb{C})}(\Gamma) \arrow[r, "1:1"] & X_{SL(2,\mathbb{C})}(\Gamma) \end{tikzcd}$$

The previous corollary shows that we can equivalently study the relationship between $SU(2)$ and $SL(2,\mathbb{C})$-representations of Γ from the point of view of their characters or from the point of view of their representations. Looking at the diagram, we also deduce

Corollary 1.2.6. *The natural inclusion $i_* : \mathcal{M}_{SU(2)}(\Gamma) \longrightarrow \mathcal{M}_{SL(2,\mathbb{C})}(\Gamma)$ is injective.*

1.3 $SU(2)$-character varieties of torus knots

We focus now on the specific case of the torus knot $G_{m,n}$ of coprime type (m,n). Henceforth, we will often denote $X_{SL(2,\mathbb{C})} = X_{SL(2,\mathbb{C})}(\Gamma)$ and omit the group in our notation. In this case

$$R_{SL(2,\mathbb{C})}(\Gamma) = \{(A,B) \in SL(2,\mathbb{C}) \mid A^m = B^n\}$$

and

$$R_{SU(2)}(\Gamma) = \{(A,B) \in SU(2) \mid A^m = B^n\}.$$

We have a decomposition of $X_{SL(2,\mathbb{C})}$

$$X_{SL(2,\mathbb{C})} = X_{red} \cup X_{irr}$$

where X_{red} is the subset of characters of reducible representations and X_{irr} is the subset of characters of irreducible representations. Inside $X_{SL(2,\mathbb{C})}$ we have $i_*(X_{SU(2)})$, i.e. the set of characters of $SU(2)$-representations. For simplicity, we will denote $Y = i_*(X_{SU(2)})$. Again, Y decomposes in $Y_{red} \cup Y_{irr}$.

Reducible representations

Proposition 1.3.1. *There is an isomorphism $Y_{red} \cong [-2,2] \subset \mathbb{R}$*

Proof. We will use, from now on, the explicit description of $X_{SL(2,\mathbb{C})}$ given in [61]. There is an isomorphism $X_{red} \cong \mathbb{C}$ given by

$$\left(A = \begin{pmatrix} t^n & 0 \\ 0 & t^{-n} \end{pmatrix}, B = \begin{pmatrix} t^m & 0 \\ 0 & t^{-m} \end{pmatrix} \longrightarrow s = t + t^{-1} \in \mathbb{C} \right)$$

This is because given a reducible $SL(2,\mathbb{C})$-representation ρ, we can consider the associated split representation $\rho = \rho' + \rho''$, for which in a certain basis takes the form:

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, B = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix},$$

and the equality $A^m = B^n$ implies that $\lambda = t^n, \mu = t^m$ for a unique $t \in \mathbb{C}$ (here we use that m, n are coprime). Now, since $A, B \in SU(2)$, t must satisfy that $|t|^2 = 1$, i.e. $t \in S^1 \subset \mathbb{C}$. We have to also take account of the change of order of the basis elements and therefore $t \sim \frac{1}{t}$. So the parameter space is isomorphic to $[-2, 2]$ (under the correspondence $t \in S^1 \longrightarrow s = t + t^{-1} = 2\operatorname{Re}(t) \in [-2, 2]$). □

To explicitly describe when a pair (A, B) is reducible, we follow [61, 2.2]. First of all, A and B are diagonalizable (recall that $A, B \in SU(2)$), so we can rule out the Jordan type case since it is not possible. So

Proposition 1.3.2. *In any of the cases:*

- $A^m = B^n \neq \pm \operatorname{Id}$
- $A = \pm \operatorname{Id}$ *or* $B = \pm \operatorname{Id}$

the pair (A, B) *is reducible.*

Proof. Let us deal with the first case, when $A^m = B^n \neq \pm \operatorname{Id}$. A is diagonalizable with respect to a basis $\{e_1, e_2\}$, and takes the form $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$. Then

$$B^n = A^m = \begin{pmatrix} \lambda^m & 0 \\ 0 & \lambda^{-m} \end{pmatrix}$$

so B is diagonal in the same basis and the pair is reducible. For the second case, if $A = \alpha \operatorname{Id}$, where $\alpha = \pm 1$, then any basis diagonalizing B diagonalizes A, hence the pair is reducible. The case $B = \alpha \operatorname{Id}$ follows in the same way. □

Irreducible representations

Now we look at the set of irreducible representations, since we want to study Y_{irr}. Let $(A, B) \in R_{SU(2)}(\Gamma)$ be an irreducible pair. Both are diagonalizable, and using Proposition 1.3.2, they must satisfy that $A^m = B^n = \pm \operatorname{Id}$, $A, B \neq \pm \operatorname{Id}$. The eigenvalues $\lambda, \lambda^{-1} \neq \pm 1$ of A satisfy $\lambda^m = \pm 1$, the eigenvalues μ, μ^{-1} of B satisfy $\mu^n = \pm 1$ and $\lambda^m = \mu^n$.

We can associate to A a basis $\{e_1, e_2\}$ under which it diagonalizes, and the same for B, obtaining another basis $\{f_1, f_2\}$. The eigenvalues λ, μ and the eigenvectors e_i, f_i completely determine the representation (A, B). We are interested in $i_*(\mathcal{M}_{SU(2)})$, $SL(2,\mathbb{C})$-equivalence classes of such pairs (A, B), and these are fully described by the projective invariant of the four points $\{e_1, e_2, f_1, f_2\}$, the cross ratio

$$[e_1, e_2, f_1, f_2] \in \mathbb{P}^1 - \{0, 1, \infty\}$$

(we may assume that the four eigenvectors are different since the representation is irreducible, see [61] for details).

Since both $A, B \in SU(2)$, we know that $e_1 \perp e_2$ and $\|e_1\| = \|e_2\| = 1$, so shifting the vectors by a suitable rotation $C \in SU(2)$, we can assume that $e_1 = [1:0], e_2 = [0:1]$, and therefore $f_1 = [a:b], f_2 = [-\bar{b} : \bar{a}]$, since they are orthogonal too. So the pair (A, B) inside $X_{SL(2,\mathbb{C})}$ is determined by λ, μ satisfying the conditions above and the projective cross ratio

$$r = \Big[e_1, e_2, f_1, f_2\Big] = \Big[0, \infty, \frac{b}{a}, -\frac{\bar{a}}{\bar{b}}\Big] = \frac{b\bar{b}}{-a\bar{a}} = \frac{b\bar{b}}{b\bar{b}-1} = \frac{t}{t-1}$$

where we have used that $a\bar{a} + b\bar{b} = 1$ and $t = |b|^2, b \in (0,1)$. We also get that r is real and $r \in (-\infty, 0)$.

The converse is also true: if the triple (λ, μ, r), satisfies that $\lambda^m = \mu^n = \pm 1$, $\lambda, \mu \neq \pm 1$ and $r \in (-\infty, 0)$, then $(A, B) \in i_*(\mathcal{M}_{SU(2)})$. To see this, r determines uniquely $t = |b|^2$ since $r(t)$ is invertible for $t \in (0,1)$. Once $|b|$ is fixed, we get that $|a|$ is fixed too, using $|a|^2 = 1 - |b|^2$. We can choose any $(a, b) \in S^1 \times S^1$ and we conclude that (A, B) is $SL(2,\mathbb{C})$-equivalent to a $SU(2)$ representation. To be more precise, it is equivalent to the representation with eigenvalues λ, μ and eigenvectors $[1:0], [0:1], [a:b], [-\bar{b}, \bar{a}]$.

Finally, we have to take account of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ action given by the permutation of the eigenvalues

- Permuting e_1, e_2 takes (λ, μ, r) to $(\lambda^{-1}, \mu, r^{-1})$.
- Permuting f_1, f_2 takes (λ, μ, r) to $(\lambda, \mu^{-1}, r^{-1})$.

Since $\lambda^m = \mu^n = \pm 1$, we get that

$$\lambda = e^{\pi i k/m}, \qquad \mu = e^{\pi i k'/n}, \tag{1.2}$$

where since $\lambda \sim \lambda^{-1}, \mu \sim \mu^{-1}$ and $\lambda \neq \pm 1, \mu \neq \pm 1$, we can restrict to the case when $0 < k < m, 0 < k' < n$. We also notice that $\lambda^m = \mu^n$ implies that $k \equiv k' \pmod 2$. So the irreducible part is made of $(m-1)(n-1)/2$ intervals.

We have just proved:

Proposition 1.3.3.

$$Y_{irr} \cong \{(\lambda, \mu, r) : \lambda^m = \mu^n = \pm 1; \lambda, \mu \neq \pm 1; r \in (-\infty, 0)\}/\mathbb{Z}_2 \times \mathbb{Z}_2$$

This real algebraic variety consists of $\frac{(m-1)(n-1)}{2}$ *open intervals.*

To describe the closure of the irreducible orbits, we have to consider the case when $e_1 = f_1$, since this is what happens in the limit (the situation is analogous when $e_2 = f_2$). In this situation $r = 0$, and the representation is equivalent to a reducible representation. Taking into account Lemma 1.3.1, it corresponds to a certain $t \in S^1$ such that $\lambda = t^n$, $\mu = t^m$. We have another limit case $r = -\infty$, if we allow $e_1 = f_2$. The representation is again reducible and corresponds to another $t' \in S^1$ such that $\lambda = (t')^n$, $\mu^{-1} = (t')^m$.

Remark 1.3.4. *The explicit description of the set of $SU(2)$-representations allows us to give an alternative proof of Corollary 1.2.6, which stated that the inclusion $i_* : \mathcal{M}_{SU(2)} \to \mathcal{M}_{SL(2,\mathbb{C})}$ is injective.*

Let us see this. Suppose that (A, B) and (A', B') are two $SU(2)$-representations which are mapped to the same point in $\mathcal{M}_{SL(2,\mathbb{C})}$, i.e. which are $SL(2,\mathbb{C})$-equivalent. If we denote by u_1, u_2, u_3, u_4 the set of eigenvectors of (A, B) and by v_1, v_2, v_3, v_4 the set of eigenvectors of (A', B'), we know that

$$[u_1, u_2, u_3, u_4] = [v_1, v_2, v_3, v_4] = r \in (-\infty, 0).$$

Since their cross ratio is the same, we know that there exists $P \in SL(2,\mathbb{C})$ that takes the set u_i to v_i. Moreover, since P takes the unitary basis u_1, u_2 to the unitary basis v_1, v_2, we get that $P \in SU(2)$, and therefore both representations are $SU(2)$-equivalent.

Topological description

We finally describe Y topologically. We refer to [61] for a geometric description of $X_{SL(2,\mathbb{C})}$.

Using proposition 1.3.3, Y_{irr} is a collection of real intervals (parametrized by $r \in (-\infty, 0)$) for a finite number of (λ, μ) that satisfy the required conditions. By our last observation, the limit cases when $r = 0, \infty$ (points in the closure of Y_{irr}) correspond to the points where the closure of Y_{irr} intersects Y_{red}.

As we saw before, each interval has two points in its closure: these are $t_0 \in S^1$ such that $t_0^n = \lambda$, $t_0^m = \mu$ ($r = 0$) and $t_1 \in S^1$ corresponding to $t_1^n = \lambda$, $t_1^m = \mu^{-1}$ ($r = -\infty$). The conditions on λ, μ force that $t_0 \neq t_1$ so that we get different intersection points with Y_{red}.

Y is topologically a closed interval (Y_{red}) with $(m-1)(n-1)/2$ closed intervals (Y_{irr}) attached at $(m-1)(n-1)$ different endpoints (without any intersections among them). The interval $Y_{red} = [-2, 2]$ sits inside $X_{red} \cong \mathbb{C}$ and every real interval in Y_{irr} is inside the corresponding complex line in X_{irr}.

The situation is described in Figures 1.1 and 1.2.

Note that in the $SU(2)$-case, since $Y_{red} \cong [-2, 2]$ is a real closed interval, we can look at the particular order of the pairs of intersection points of the closure of Y_{irr} with Y_{red}. This

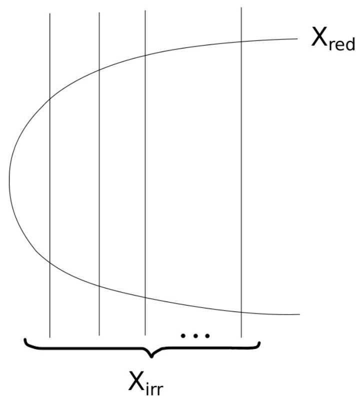

Figure 1.1: Picture of $X_{SL(2,\mathbb{C})}$, defined over $\mathbb{C}$. The drawn lines are curves isomorphic to $\mathbb{C}$. The closure of each curve in X_{irr} intersects X_{red} at two distinct points.

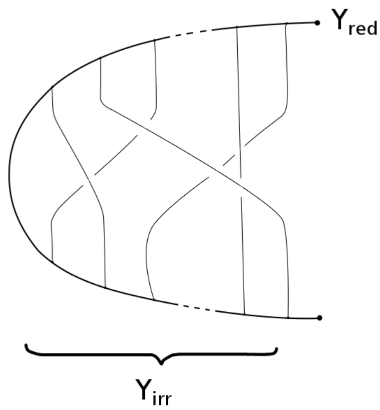

Figure 1.2: Picture of $Y \subset X_{SL(2,\mathbb{C})}$, defined over $\mathbb{R}$. The picture displays the set of real segments which form Y_{irr}.

is why the above picture displays Y_{red} as a collection of tangled intervals, in contrast to the $SL(2,\mathbb{C})$-case where no ordering can be defined.

More concretely, each component of Y_{irr} is characterized by a triple (λ,μ,r), where $\lambda = e^{\frac{\pi i k}{m}}$, $\mu = e^{\frac{\pi i k'}{n}}$, $0 < k < m$, $0 < k' < n$ and $k \equiv k' \pmod 2$ (cf. Proposition 1.3.3). Its closure intersects Y_{red} at two points: the two reducible representations described by the eigenvalues (λ,μ) and (λ,μ^{-1}). There is a unique t_1 such that $t_1^n = \lambda$, $t_1^m = \mu$, and a unique t_2 such that $t_2^n = \lambda$, $t_2^m = \mu^{-1}$. The points $s_i = t_i + t_i^{-1} \in [-2,2]$ give us the intersection points with $Y_{red} \cong [-2,2]$. Since both t_i are n-th roots of λ, they will be of the form

$$t_i = e^{\frac{\pi i(k+2a_i m)}{mn}}$$

for certain a_i verifying $0 \leq a_i < n$. Solving the equation $t_1^m = \mu$ and $t_2^m = \mu^{-1}$, we get that a_1, a_2 are the unique solutions to the equations:

$$k + 2a_1 m \equiv k' \pmod{2n}$$

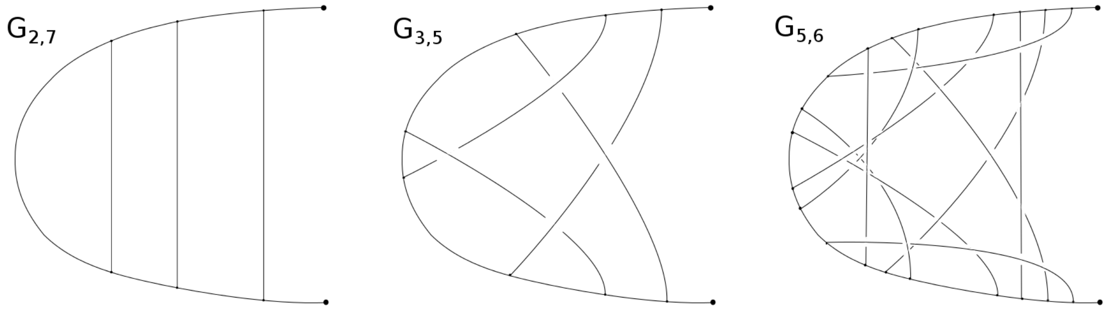

Figure 1.3: Examples of several character varieties for some $G_{m,n}$.

$$k + 2a_2m \equiv 2n - k' \pmod{2n}$$

We finally obtain that the intersection points of the component given by the triple (k, k', r) are the points

$$s_1 = 2\cos\left(\frac{\pi k}{mn} + \frac{2a_1\pi}{n}\right) \quad s_2 = 2\cos\left(\frac{\pi k}{mn} + \frac{2a_2\pi}{n}\right).$$

The ordering of these sets of pairs of points (one pair for each admissible (k, k')) depends on the type of torus knot group, i.e. on (m, n). As Figure 1.3 shows, we can obtain all kind of situations depending on the particular choice of (m, n). Looking at $G_{5,6}$, notice that it is not always true that we always have pairs of positive and negative endpoints.

A natural question is whether the inclusion of the $SU(2)$-character variety Y inside the $SL(2, \mathbb{C})$-character variety is a homotopy equivalence, i.e., if the two varieties have the same homotopy type. The result is in general false if we choose an arbitrary finitely generated group Γ, but remains true in some cases, for example $\Gamma = \mathbb{Z}^k$ see [68].

In general, given G a complex reductive algebraic group and K a maximal compact subgroup, there are some cases when $\mathrm{Hom}(\Gamma, K)//K$ is a deformation retract of $\mathrm{Hom}(\Gamma, G)//G$, as above. For example, this occurs when Γ is a free group [24], a finitely generated abelian group [26] or when Γ is nilpotent [4]. A remarkable counterexample is given in [5] when Γ is a surface group: in that case, the character variety $\mathrm{Hom}(\Gamma, K)/K$ is homeomorphic to the moduli space of topologically trivial semistable principal G-bundles on X, whereas $\mathrm{Hom}(\Gamma, G)//G$ is homeomorphic to the moduli space of semistable G-Higgs bundles (E_G, Φ) on X such that E_G is topologically trivial. Cohomology computations show that these two spaces are not homeomorphic, showing that there cannot exist a retraction between them. Recent articles have studied the case where G is real reductive [11], and the case when Γ is a virtually nilpotent Kähler group [6].

Looking at the explicit description of Y and $X_{SL(2,\mathbb{C})}$, we obtain, in our case,

Corollary 1.3.5. *Y is a reformation retract of $X_{SL(2,\mathbb{C})}$.*

1.4 Noncoprime case

If $\gcd(m,n) = d > 1$, then $G_{m,n}$ does no longer represent a torus knot, since these are only defined in the coprime case. However, the group $G_{m,n} = \langle x, y \mid x^n = y^m \rangle$ still makes sense and we can study the representations of this group into $SL(2,\mathbb{C})$ and $SU(2)$ using the method described above. We will denote by a, b the integers that satisfy

$$\begin{aligned} m &= a\,d, \\ n &= b\,d. \end{aligned}$$

As we did before, we focus on $Y = i_*(X_{SU(2)})$, the set of characters of $SU(2)$-representations.

Reducible representations

First of all, we describe what happens in the $SL(2,\mathbb{C})$ case.

Proposition 1.4.1. *There is an isomorphism*

$$X_{red} \cong \bigsqcup_{i=0}^{\lfloor d/2 \rfloor} X^i_{red}$$

where:

- *$X^i_{red} \cong \mathbb{C}^*$ for $0 < i < \frac{d}{2}$.*
- *$X^i_{red} \cong \mathbb{C}$ for $i = 0$ and $i = \frac{d}{2}$ if d is even*

Proof. As it is shown in [61], an element in X_{red} can be regarded as the character of a split representation, $\rho = \rho' \oplus \rho'^{-1}$. There is a basis such that

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad B = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix},$$

where $A^m = B^n$ implies that $\lambda^m = \mu^n$. We deduce that $(\lambda^a)^d = (\mu^b)^d$, so that (λ, μ) belong to one of the components

$$X^i_{red} = \{(\lambda,\mu) | \lambda^a = \xi^i \mu^b\} = \{(\lambda,\mu) | \lambda^a \mu^{-b} = \xi^i\},$$

where ξ is a primitive d-th root of unity. These components are disjoint, and each one of them is parametrized by $\mathbb{C}^*$. To see this, let us fix a component, X^i_{red}, and let α be a b-th root of ξ^i. Then

$$\begin{aligned} X^i_{red} &= \{(\lambda,\mu) | \lambda^a = \xi^i \mu^b\} \\ &= \{(\lambda,\mu) | \lambda^a = \alpha^b \mu^b\} \\ &= \{(\lambda,\nu) | \lambda^a = \nu^b\} \cong \mathbb{C}^*\,. \end{aligned}$$

In other words, for each $(\lambda,\mu)\in X^i_{red}$ there is a unique $t\in\mathbb{C}^*$ such that $t^b=\lambda$, $t^a=\alpha\mu$. However, we have to take account of the action given by permuting the two vectors in the basis, which corresponds to the change $(\lambda,\mu)\sim(\lambda^{-1},\mu^{-1})$. In our decomposition, if $(\lambda,\mu)\in X^i_{red}$, then $(\lambda^{-1},\mu^{-1})\in X^{-i}_{red}$. So $t\in X^i_{red}$ is equivalent to $1/t\in X^{-i}_{red}$.

For $0\le i\le d-1$, we have two possibilities. If $i\not\equiv -i \pmod d$, then X^i_{red} and X^{-i}_{red} get identified. If $i\equiv -i \pmod d$, then $t\sim t^{-1}\in X^i_{red}\cong\mathbb{C}$, and thus $X^i_{red}/_\sim\cong\mathbb{C}^*/_{a\sim a^{-1}}\cong\mathbb{C}$.

When d is even, there are two $i\in\mathbb{Z}/d\mathbb{Z}$ such that $i\equiv -i \pmod d$, so we get two copies of $\mathbb{C}$ in Y_{red}. When d is odd we get just one, since there is only one solution ($i\equiv 0$). The remaining copies of X^i_{red} get identified pairwise: $X^i_{red}\sim X^{-i}_{red}$. □

Now, for the case of $SU(2)$-representations, we have

Proposition 1.4.2. *There is an isomorphism*

$$Y_{red}\cong\bigsqcup_{i=0}^{\lfloor\frac{d}{2}\rfloor}Y^i_{red}$$

, where:

- $Y^i_{red}\cong S^1$ *for* $0<i<\frac{d}{2}$
- $Y^i_{red}\cong[-2,2]$ *for* $i=0$, $i=\frac{d}{2}$ *if* d *is even*

Proof. If (A,B) is a reducible $SU(2)$-representation, both are diagonalizable with respect to a certain basis and therefore

$$A=\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix},\quad B=\begin{pmatrix}\mu & 0\\ 0 & \mu^{-1}\end{pmatrix}.$$

The equality $A^m=B^n$ gives us that $\lambda^m=\mu^n$. So the pair (λ,μ) belongs to a certain component X^i_{red}. Since it is a $SU(2)$-representation, the eigenvalues λ and μ satisfy that $|\lambda|=|\mu|=1$. This implies that $(\lambda,\mu)\in S^1\subset\mathbb{C}^*\cong X^i_{red}$: we define $Y^i_{red}:=S^1\subset X^i_{red}$.

We have to take into account the equivalence relation in X_{red} given by the permutation of the eigenvectors. If $i\not\equiv -i \pmod d$, then $Y^i_{red}\cong Y^{-i}_{red}$. If $i\equiv -i \pmod d$, then $Y^i_{red}\cong S^1/_{a\sim a^{-1}}\cong[-2,2]$. This gives the desired result. □

Irreducible representations

We start by describing what happens in the $SU(2)$ case.

Proposition 1.4.3. *We have an isomorphism*

$$Y_{irr}\cong\{(\lambda,\mu,r):\lambda^m=\mu^n=\pm1;\lambda,\mu\neq\pm1,r\in(-\infty,0)\}/\mathbb{Z}_2\times\mathbb{Z}_2\,.$$

This real algebraic variety consists of:

- $\frac{(m-1)(n-1)+1}{2}$ *open intervals if* m, n *are both even,*
- $\frac{(m-1)(n-1)}{2}$ *open intervals in any other case.*

Proof. By Proposition 1.3.2, a representation (A, B) is reducible unless $A^m = B^n = \pm\,\mathrm{Id}$, $A, B \neq \pm\,\mathrm{Id}$. So the set of irreducible representations can be described using the same tools as before: the set of equivalence classes of irreducible representations is a collection of intervals $r \in (-\infty, 0)$ parametrized by pairs (k, k') satisfying

$$0 < k < m, \quad 0 < k' < n, \quad k \equiv k' \pmod 2. \tag{1.3}$$

We compute the number of such pairs, separating in three different cases according to the parity of m and n:

Suppose m, n are both even. If $k \equiv k' \equiv 0 \pmod 2$, then $k \in \{2, 4, \dots, m-2\}$, $k' \in \{2, 4, \dots n-2\}$, so there are $\frac{(m-2)(n-2)}{4}$ such pairs. If $k \equiv k' \equiv 1 \pmod 2$, $k \in \{1, 3, \dots m-1\}$, $k' \in \{1, 3, \dots, n-1\}$, we have $\frac{mn}{4}$ pairs. The sum is $\frac{(m-2)(n-2)}{4} + \frac{mn}{4} = \frac{(m-1)(n-1)+1}{4}$.

Suppose m is even and n is odd (the case m odd and n even is similar). Then if $k \equiv k' \equiv 0 \pmod 2$, $k \in \{2, 4, \dots, m-2\}$, $k' \in \{2, 4, \dots n-1\}$, we get $\frac{(m-2)(n-1)}{4}$ such pairs. If $k \equiv k' \equiv 1 \pmod 2$, $k \in \{1, 3, \dots m-1\}$, $k' \in \{1, 3, \dots, n-2\}$, and there are $\frac{m(n-1)}{4}$ such pairs. We get in total $\frac{m(n-1)}{4} + \frac{(m-2)(n-1)}{4} = \frac{(m-1)(n-1)}{2}$.

Finally, suppose both m, n odd. If $k \equiv k' \equiv 0 \pmod 2$, $k \in \{2, 4, \dots, m-1\}$, $k' \in \{2, 4, \dots n-1\}$, and we get $\frac{(m-1)(n-1)}{4}$ such pairs. If $k \equiv k' \equiv 1 \pmod 2$, $k \in \{1, 3, \dots m-2\}$, $k' \in \{1, 3, \dots, n-2\}$, there are $\frac{(m-1)(n-1)}{4}$ such pairs. We get $\frac{(m-1)(n-1)}{2}$ pairs in total.

We have obtained a decomposition

$$Y_{irr} = \bigsqcup_{k,k'} Y_{irr}^{(k,k')}$$

where every $Y_{irr}^{(k,k')}$ is an open interval isomorphic to $(-\infty, 0)$. ☐

For the case of $SL(2, \mathbb{C})$-representations, we have the following,

Proposition 1.4.4. *The component* $X_{irr} \subset X_{SL(2,\mathbb{C})}$ *is described as*

$$X_{irr} = \bigsqcup_{k,k'} X_{irr}^{(k,k')}$$

where k, k' *satisfy* (1.3), *and* $X_{irr}^{(k,k')} = \mathbb{P}^1 - \{0, 1, \infty\}$. *This complex algebraic variety consists of* $\frac{(m-1)(n-1)+1}{2}$ *components if* m, n *are both even, of* $\frac{(m-1)(n-1)}{2}$ *components if one of* m, n *is odd. Moreover* $Y_{irr}^{(k,k')} = (-\infty, 0) \subset X_{irr}^{(k,k')}$ *in the natural way.*

The limit cases $r = 0$, $r = -\infty$ correspond to the closure of the irreducible components, and these points are exactly where $\overline{Y}_{irr}$ intersects Y_{red}. The triples $(\lambda, \mu, 0)$, $(\lambda, \mu, -\infty)$ correspond to the reducible representations with eigenvalues (λ, μ) and (λ, μ^{-1}). Since $\lambda, \mu \neq \pm 1$, we get two different intersection points. Note that the pattern of intersections for $\overline{X}_{irr}$ and X_{red} is the same, but the components are complex algebraic varieties now.

In order to understand the way the closure of the components of Y_{irr} intersect Y_{red},

Proposition 1.4.5. *The closure of $Y_{irr}^{(k,k')}$ is a closed interval that joins $Y_{red}^{i_0}$ with $Y_{red}^{i_i}$, where*

$$i_0 = \frac{k-k'}{2}, \quad i_i = \frac{k+k'}{2} \pmod{d}\,.$$

Proof. Set $D = 2dab$, and consider ω a primitive D-th root of unity. Then $\xi := \omega^{D/d} = \omega^{2ab}$ is a primitive d-th root of unity. The irreducible component $Y_{irr}^{(k,k')}$ is the interval (λ, μ, r), $r \in (-\infty, 0)$, where

$$\lambda = (\omega^b)^k, \quad \mu = (\omega^a)^{k'},$$

and k, k' are subject to the conditions (1.3), see equation (1.2). The points in the closure of $Y_{irr}^{(k,k')}$ correspond to the reducible representations with eigenvalues (λ, μ) and (λ, μ^{-1}). Clearly $(\lambda, \mu) \in X_{red}^{i_0}$, since

$$\lambda^a \mu^{-b} = \omega^{kab} \omega^{-k'ab} = \omega^{\frac{k-k'}{2}2ab} = \omega^{i_0 2ab} = \xi^{i_0}\,,$$

and $(\lambda, \mu^{-1}) \in X_{red}^{i_1}$, since

$$\lambda^a \mu^b = \omega^{kab} \omega^{k'ab} = \xi^{i_1}\,.$$

□

Proposition 1.4.5 gives a clear rule to depict $Y = Y_{irr} \cup Y_{red}$ for every pair (m, n). Acutally, Y is a collection of intervals attached on their endpoints to Y_{red}, which consists of several disjoint copies of S^1 and $[-2, 2]$. Note that the pattern of intersections for the irreducible components of $X_{SL(2,\mathbb{C})} = X_{irr} \cup X_{red}$ is the same as that of Y.

When m, n are coprime, we recover our previous pictures.

Corollary 1.4.6. *For any two different components $Y_{red}^{i_0}, Y_{red}^{i_1} \subset Y_{red}$, there is a pair (k, k') such that $\overline{Y}_{irr}^{(k,k')}$ joins them.*

In particular, Y is a connected topological space.

Proof. We can assume $0 \leq i_0 < i_i \leq \frac{d}{2}$. Then $0 < k = d + i_0 - i_1 < d \leq m$ and $0 < k' = d - i_0 - i_1 < d \leq n$ both satisfy that $k \equiv k' \pmod 2$ and $\frac{k-k'}{2} = i_0$, $\frac{k+k'}{2} = i_1$. □

Remark 1.4.7. *It can be checked that there is no component* $\overline{Y}_{irr}^{(k,k')}$ *which joins* $Y_{red}^{i_0}$ *to itself when* $m = n$*, or when one of* m, n *divides the other, and we are dealing with* $i_0 = 0$ *or* $i_0 = d/2$ *(the latter only if* d *is even).*

Actually, such component would correspond to a pair (k, k') *such that* $\frac{k-k'}{2} \equiv \pm i_0 \pmod d$ *and* $\frac{k+k'}{2} \equiv \pm i_0 \pmod d$*. Accounting for all possibilities of signs, we have either* $k \equiv \pm 2i_0$*,* $k' \equiv 0 \pmod d$*, or* $k \equiv 0$*,* $k' \equiv \pm 2i_0 \pmod d$*. This has solutions unless* $m > n = d$*,* $i_0 = 0, d/2$*;* $n > m = d$*,* $i_0 = 0, d/2$*; or* $m = n = d$*, any* i_0*.*

Finally, as it happened in the coprime case,

Corollary 1.4.8. Y *is a reformation retract of* $X_{SL(2,\mathbb{C})}$*.*

Proof. We see, looking at Propositions 1.4.1 and 1.4.2, that each component of Y_{red}, which is either isomorphic to $[-2, 2]$ or S^1, is a deformation retract of its corresponding component in X_{red} (isomorphic to $\mathbb{C}$ or $\mathbb{C}^*$, respectively). Besides, the closure of each component in Y_{irr}, isomorphic to $[0, \infty]$, is again a deformation retract of the closure of its corresponding component in X_{irr} (isomorphic to $\mathbb{C}$). Using the gluing lemma, we can construct a global homotopy to show that Y is a deformation retract of X, as desired. $\square$

Chapter 2

E-polynomials

2.1 Introduction. Mixed Hodge structures

We start by reviewing some basic Hodge theory. We refer to [80] for further details. Given X a complex manifold, there are local complex charts $\varphi : U \to \mathbb{C}^n$, $\varphi(u) = (z_1, \dots, z_n)$, where $U \subset X$ is an open subset and $z_i = x_i + iy_i$. The set of real coordinates $(x_1, y_1, \dots, x_n, y_n)$ is a real chart for the underlying differentiable manifold. Therefore, every 1-form $\alpha \in \Omega^1(X, \mathbb{R})$ can be expressed in terms of the differentials $dx_1, dy_1, \dots, dx_n, dy_n$. If we look at complex valued 1-forms, which are elements in $\Omega^1(X, \mathbb{C}) := \Omega^1(X, \mathbb{R}) \otimes \mathbb{C}$, we can write any of them as a combination of the 1-forms

$$dz_j = dx_j + idx_j \quad d\overline{z}_j = dx_j - idy_j \quad j = 1 \dots n.$$

A (p, q)*-form* is a smooth complex-valued differential k-form β, where $k = p + q$, that can be written as

$$\beta = \sum f_{i_1,\dots,i_p,j_1,\dots,j_q}(z) dz_{i_1} \wedge \dots \wedge dz_{i_p} \wedge d\overline{z}_{j_1} \wedge \dots \wedge d\overline{z}_{j_q}$$

Using multi-indexes, we can write $I = \{i_1, \dots, i_p\}$, $J = \{j_1, \dots, j_p\}$ and $dz_I = dz_{i_i} \wedge \dots \wedge dz_{i_p}, d\overline{z}_J = d\overline{z}_{j_1} \wedge \dots \wedge d\overline{z}_{j_q}$, so that

$$\beta = \sum_{|I|=p, |J|=q} f_{IJ} dz_I \wedge d\overline{z}_J.$$

Let us assume that X is a compact Kähler manifold. Let $H^{p,q}(X)$ denote the set of cohomology classes in $H^k(X, \mathbb{C})$ that can be represented by differential forms of type (p, q). A classical theorem of Hodge asserts

Theorem 2.1.1 (Hodge decomposition). *Let X be a compact complex Kähler manifold of complex dimension m. There is a direct sum decomposition*

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$$

for each $k = 1, \ldots, 2m$, *such that* $\overline{H^{p,q}} = H^{q,p}$.

Smoothness is required in order to work with differential forms, compactness is needed to use elliptic operator theory applied to the Laplacian and the Kähler identities play an important role. Hodge decomposition asserts that every cohomology class has a unique harmonic representative, and that the (p, q)-bigrading is preserved by the Laplace operator. This leads to the modern definition of a pure Hodge structure.

Definition 2.1.2. *A* pure Hodge structure *of weight* k *consists of a finite dimensional complex vector space* H *with a real structure, and a decomposition*

$$H = \bigoplus_{k=p+q} H^{p,q}$$

such that $H^{q,p} = \overline{H^{p,q}}$, *the bar meaning complex conjugation on* H.

We note that in the literature a pure Hodge structure is often defined over a rational or real vector space H, but for our purposes, a complex vector space with a fixed real structure will suffice.

A pure Hodge structure of weight k gives rise to the so-called Hodge filtration, which is a descending filtration $F^\bullet$, where

$$F^p = \bigoplus_{s \geq p} H^{s,k-s}.$$

We can define $\mathrm{Gr}^p_F(H) := F^p/F^{p+1} = H^{p,k-p}$ and stablish in this way the equivalence between both notions, the graded and the filtered one. Since both concepts coincide, a pure Hodge structure of weight k is often defined as follows

Definition 2.1.3. *A* pure Hodge structure *of weight* k *on a complex vector space* H *with a real structure is a descending filtration*

$$F^0 = H \supset F^1 \supset \ldots \supset \{0\}$$

such that $F^p \cap \overline{F^{k-(p-1)}} = \{0\}$ *for all* p.

We can rephrase Theorem 2.1.1 by saying that the cohomology groups of a complex compact Kähler manifold admit a pure Hodge structure.

A natural question to ask is whether it is possible to extend this notion to other situations. Is there an analogous notion to pure Hodge structures for other spaces, possibly singular or non-compact? Do algebraic varieties admit such filtrations? The work of Deligne [17, 18] stablished the foundations of modern Hodge theory. We will give the necessary definitions and some properties of mixed Hodge structures. A detailed treatment and complete proofs can be found in [67].

Definition 2.1.4. *A* mixed Hodge structure *consists of a finite dimensional complex vector space H with a real structure and:*

- (Weight filtration) *An ascending filtration $W_\bullet$,*

$$0 = W_{-1} \subset W_0 \subset \ldots \subset W_{k-1} \subset W_k \subset \ldots \subset W_{2k} = H$$

- (Hodge filtration) *A descending filtration $F^\bullet$,*

$$F^0 = H \supset F^1 \supset \ldots \supset \{0\}$$

such that $F^\bullet$ induces a pure Hodge structure on each graded piece $Gr_l = W_l/W_{l-1}$. We define

$$H^{p,q} := \operatorname{Gr}_F^p \operatorname{Gr}_{p+q}^W(H)$$

and write $h^{p,q}$ for the Hodge number $h^{p,q} := \dim_{\mathbb{C}} H^{p,q}$.

Deligne showed in [17] that

Theorem 2.1.5. *Let X be an quasi-projective algebraic variety. Then the cohomology groups $H^k(X, \mathbb{C})$ of X are endowed with a mixed Hodge structure for every k.*

Intuitively, each cohomology group $H^k(X, \mathbb{C})$ does not admit a pure Hodge structure, but we can filter it in such a way that its graded pieces, Gr_l, look like the l-th cohomology group of a smooth projective variety. Compactly supported cohomology can be also endowed with a mixed Hodge structure. Therefore, if X is a quasi-projective algebraic variety (maybe non-smooth or non-compact), we define the *Hodge numbers* of X by

$$h_c^{k,p,q}(X) = h^{p,q}(H_c^k(X)) = \dim_{\mathbb{C}} \operatorname{Gr}_F^p \operatorname{Gr}_{p+q}^W H_c^k(X).$$

We list some some properties of mixed Hodge structures in the next proposition.

Proposition 2.1.6. *Let X be a quasi-projective variety. Then:*

1. *Mixed Hodge structures are functorial: for any algebraic map $f : X \mapsto Y$ between algebraic varieties,*

$$f^*(W_m) \subset W_m$$

   Actually, f^* preserves the pure Hodge structure of weight m on the graded pieces Gr_m for each m. An important consequence is that a short exact sequence of cohomology groups induced by algebraic maps remains exact after taking Gr_m for each m.

2. *If X is smooth and projective (hence Kähler and compact), its mixed Hodge structure is actually pure. In other words, the weight filtration satisfies*

$$0 = W_{m-1} \subset W_m = H^m(X)$$

for each m.

3. *If X is smooth (but not necessarily projective), then*

$$0 = W_{m-1} \subset W_m \subset \ldots \subset W_{2m} = H^m(X)$$

4. *If X is projective (but not necessarily smooth), then*

$$0 = W_{-1} \subset W_0 \subset \ldots W_m = H^m(X)$$

Note that, in terms of Hodge numbers and passing to compactly supported cohomology, Properties 3 and 4 imply that $h_c^{k,p,q} = 0$ for $p + q > k$ when X is smooth and $h_c^{k,p,q} = 0$ for $p + q < k$ when X is projective.

2.2 E-polynomials

There are two important polynomials that can be defined using the Hodge numbers of any algebraic variety X.

Definition 2.2.1. *The* mixed Hodge polynomial*, $h(X) \in \mathbb{Z}[u, v, t]$ is defined as*

$$h(X)(u, v, t) = \sum_{p,q,k} h_c^{k,p,q}(X) u^p v^q t^k$$

Definition 2.2.2. *The Hodge-Deligne polynomial, or* E-polynomial*, $e(X) \in \mathbb{Z}[u, v]$ is defined as*

$$e(X) = e(X)(u, v) := \sum_{p,q,k} (-1)^k h_c^{k,p,q}(X) u^p v^q.$$

Alternatively, if we write $\chi^{p,q}(X) = \sum_k (-1)^k h_c^{k,p,q}(X)$, then

$$e(X) = \sum_{p,q} \chi^{p,q} u^p v^q.$$

It is clear that the E-polynomial is a specialization of the mixed Hodge polynomial at $t = 1$: $e(X)(u, v) = h(X)(u, v, -1)$, so it carries less information than the mixed Hodge polynomial. They agree when the Hodge structure is pure.

Remark 2.2.3. *When $h_c^{k,p,q} = 0$ for $p \neq q$, the polynomial $e(Z)$ depends only on the product uv. The mixed Hodge structure is said to be of Hodge-Tate type. This will happen in all the cases that we shall investigate here. In this situation, it is conventional to use the variable $q = uv$ for $e(X)$, so $e(X) \in \mathbb{Z}[q]$. If this happens, we will also say that the variety is of balanced type. For instance, $e(\mathbb{C}^n) = q^n$.*

One of the advantages of E-polynomials that makes them easier to compute is that they are additive for stratifications of X.

Proposition 2.2.4. *If Z is a complex algebraic variety and $Z = \bigsqcup_{i=1}^n Z_i$, where all Z_i are locally closed in Z, then*

$$e(Z) = \sum_{i=1}^{n} e(Z_i).$$

Proof. It suffices to prove that if Z is a complex quasi-projective variety, Y is a closed subvariety and $U = Z - Y$, we have $e(Z) = e(Y) + e(U)$.

If we look at the long exact sequence of cohomology with compact support

$$\ldots \to H_c^k(U) \to H_c^k(Z) \to H_c(Y) \to H_c^{k+1}(U) \to \ldots$$

the maps in the sequence are compatible with the weight and Hodge filtrations. Applying Property 1 in Proposition 2.1.6, the induced sequence on the (p,q)-pieces is still exact for all (p,q),

$$\ldots Gr_F^p Gr_{p+q}^W H_c^k(U) \to Gr_F^p Gr_{p+q}^W H_c^k(Z) \to Gr_F^p Gr_{p+q}^W H_c^k(Y) \to Gr_F^p Gr_{p+q}^W H_c^{k+1}(U) \to \ldots$$

Therefore, taking Euler characteristics, $\chi^{p,q}(Z) = \chi^{p,q}(Y) + \chi^{p,q}(U)$, from where we deduce that $e(Z) = e(Y) + e(U)$. □

Remark 2.2.5. *From the definition of the E-polynomial,*

$$e(1,1) = \chi(X),$$

the Euler characteristic of X. Also, assume that X is smooth and projective. Then

$$e(-y,1) = \sum_{p,q} (-1)^q h^{p,q} y^p = \chi_y(X),$$

which is the Hirzebruch χ_y-genus of X [46].

The leading coefficient of the E-polynomial $e(X)$ also contains additional information.

Proposition 2.2.6. *Let X be an algebraic variety. Then $e(X)(u,v)$ is a polynomial of degree $2\dim_{\mathbb{C}} X$ and the term of highest degree is $m(uv)^{\dim_{\mathbb{C}} X}$, where m is the number of irreducible components of maximal dimension.*

Proof. Let us write $n = \dim_{\mathbb{C}} X$. If X is smooth and projective, then Poincare duality gives that the leading coefficient of $e(X)$ is $h_c^{2n,n,n} = h^{0,0,0}$, the number of connected components of X, which is equal to m in this case.

Now, let $X = X_1 \cup \ldots \cup X_m$ be the decomposition of X into irreducible components. Let $U_i = X_i \setminus \bigcup_{j \neq i} X_i$, and $U = \bigcup_{i=1}^{m} U_i$. We have that $\dim(X \setminus U) < \dim X$ and that $e(X) = e(U) + e(X \setminus U)$. By dimension induction, it suffices to prove that if X is an irreducible variety, the leading coefficient of $e(X)$ is 1.

Let X be irreducible algebraic variety. By Hironaka's theorem on the resolution of singularities, there exists a birrational morphism $f : Y \longrightarrow X$ with Y nonsingular, connected and projective. Since they are birrational, there are isomorphic open subsets $U \subset X, V \subset Y$ such that $\dim(X \setminus U)$, $\dim(Y \setminus V) < \dim X$. Clearly, $e(X) = e(U) + e(X \setminus U) = e(V) + e(X \setminus U)$. The result follows by dimensional induction, since $\dim(X \setminus U) < \dim X$ and the leading coefficient of $e(V)$ is equal to the leading coefficient of $e(Y)$, which is one since Y is irreducible, smooth and projective. □

Remark 2.2.7. *These properties can be rephrased in a different setting. Let $K_0(Var/\mathbb{C})$ be the Grothendieck group of varieties over $\mathbb{C}$: the free abelian group on the set of isomorphism classes of varieties over $\mathbb{C}$, by relations of the form*

$$[X] = [Y] + [X \setminus Y]$$

where Y is a closed subvariety of X. It is a commutative ring, with product given by $[X].[Y] = [(X \times Y)]$, the fibered product defined over Spec $\mathbb{C}$. The E-polynomial is a ring homomorphism

$$e : K_0(Var/\mathbb{C}) \longrightarrow \mathbb{Z}[u, v]$$

$K_0(Var/\mathbb{C})$ is in fact generated by classes of nonsingular, connected projective varieties over $\mathbb{C}$, giving an alternative proof of Proposition 2.2.6.

2.3 Fibrations

We are interested in the behaviour of the E-polynomial under fibrations. More concretely, we will have to deal with fibrations

$$F \longrightarrow Z \xrightarrow{\pi} B, \tag{2.1}$$

where F, Z, B are quasi-projective varieties, that are locally trivial in the analytic topology. We want to compute the E-polynomial of the total space Z in terms of the E-polynomials of the base and the fibre.

Fibration (2.1) defines a local system $\mathcal{H}_c^k$, whose fibres are the cohomology groups $H_c^k(F_b)$, where $b \in B$, $F_b = \pi^{-1}(b)$. The fibres possess mixed Hodge structures, and the subspaces $W_t(H_c^k(F_b))$ are preserved by the holonomy. Since it will happen in all the cases we will be dealing with, we will assume from now on that F is of balanced type, so $\mathrm{Gr}_{2p}^W H_c^k(F_b) = H_c^{k,p,p}(F_b)$. Associated to the fibration, there is a monodromy representation

$$\rho : \pi_1(B) \longrightarrow \mathrm{GL}(H_c^{k,p,p}(F)) \tag{2.2}$$

for every k, p. Suppose that the monodromy group $\Gamma = \mathrm{im}\,(\rho)$ is an abelian and finite group. Then $H_c^{k,p,p}(F)$ are modules over the representation ring $R(\Gamma)$.

Definition 2.3.1. *The* Hodge monodromy representation *is defined as the polynomial with coefficients in the representation ring*

$$R(Z) := \sum (-1)^k H_c^{k,p,p}(F)\, q^p \in R(\Gamma)[q]\,. \tag{2.3}$$

As the monodromy representation (2.2) has finite image, there is a finite covering $B_\rho \to B$ such that the pull-back fibration

$$\begin{tikzcd} Z' \arrow[r] \arrow[d, "\pi'"'] & Z \arrow[d, "\pi"'] \\ B_\rho \arrow[r] & B \end{tikzcd} \tag{2.4}$$

has trivial monodromy.

Let $S_1, \ldots, S_N$ be the irreducible representations of Γ (there are $N = \#\Gamma$ of them, and all of them are one-dimensional). These are generators of $R(\Gamma)$ as a free abelian group. We write the Hodge monodromy representation of (2.3) as

$$R(Z) = a_1(q)S_1 + \ldots a_N(q)S_N.$$

Theorem 2.3.2. *Suppose that B_ρ is of balanced type. Then Z is of balanced type. Moreover, there are polynomials $s_1(q), \ldots, s_N(q) \in \mathbb{Z}[q]$ (only dependent on B, B_ρ and Γ, but not on the fibration or the fibre) such that*

$$e(Z) = a_1(q)s_1(q) + \ldots + a_N(q)s_N(q),$$

for $R(Z) = a_1(q)S_1 + \ldots a_N(q)S_N$.

Proof. The Leray spectral sequence of the fibration (2.1) has E_2-term

$$E_2^{l,m}(Z) = H_c^l(B, H_c^m(F)) \tag{2.5}$$

and abuts to $H_c^k(Z)$. By [1], $E_j^{l,m}(Z)$ has a mixed Hodge structure for $j \geq 2$, and the differentials d_j are compatible with the mixed Hodge structure. Therefore $e(Z) = e(H_c^*(Z)) = e(E_2^{*,*}(Z))$.

The mixed Hodge structure associated to π' in (2.4) is the product mixed Hodge structure $E_2^{l,m}(Z') = H_c^l(B_\rho) \otimes H_c^m(F)$. By our assumption, B_ρ and F are of balanced type, so Z' is of balanced type. There is a map $E_2^{l,m}(Z') \to E_2^{l,m}(Z)$, which preserves the mixed Hodge structures. This map is surjective, so Z is of balanced type.

By definition,

$$R(Z) = \sum (-1)^m H_c^{m,p,p}(F)\, q^p = a_1(q)S_1 + \ldots + a_N(q)S_N\,.$$

The local systems $S_i \to B$ are 1-dimensional and have a mixed Hodge structure. When we pull-back $S_i \to B$ to B_ρ, we get trivial local systems. Hence there are surjections $H_c^l(B_\rho) \twoheadrightarrow H_c^l(B, S_i)$, such that the image has the induced mixed Hodge structure. So $e(H_c^*(B, H_c^*(F))) = e(H_c^*(B, R(Z))) = \sum a_i(q) e(H_c^*(B, S_i))$. It suffices to define $s_i(q) = e(H_c^*(B, S_i))$ to get the statement. □

Write $e(S_i) = s_i(q)$, $1 \leq i \leq N$. Theorem 2.3.2 implies that there is a $\mathbb{Z}[q]$-linear map

$$e : R(\Gamma)[q] \to \mathbb{Z}[q]$$

satisfying the property that $e(R(Z)) = e(Z)$. We state some corollaries that derive from this fact.

Corollary 2.3.3. *Assume that $\pi : Z \to B$ is a fibre bundle with fibre F such that the action of $\pi_1(B)$ on $H_c^*(F)$ is trivial (there is no monodromy). Then*

$$e(Z) = e(B)e(F)$$

Proof. Under the hypothesis, $R(Z) = e(F)T$, where T is the trivial local system. Since $e(T) = e(B)$ (apply it to the trivial fibration $B \to B$), Theorem 2.3.5 gives the desired result (an alternative proof appears in [53, Proposition 2.4]). □

Remark 2.3.4. *The hypothesis that the action of $\pi_1(B)$ on $H_c^*(F)$ is trivial holds in particular in the following cases:*

- *B is irreducible and π is locally trivial in the Zariski topology.*
- *π is a principal G-bundle with G a connected algebraic group.*
- *Z is a G-space with isotropy $H < G$ such that $G/H \to Z \to B$ is a fibre bundle, and G is a connected algebraic group.*

We can also recover easily from Theorem 2.3.5 the result in [53, Proposition 2.10].

Corollary 2.3.5. *Let $B = \mathbb{C} - \{q_1, \dots, q_\ell\}$. Suppose that B_ρ is a rational curve. Then*

$$e(Z) = (q-1)\, e(F)^{inv} - (\ell-1)e(F),$$

where $e(F)^{inv}$ denotes the E-polynomial of the invariant part of the cohomology of F.

Proof. Let Γ be the monodromy, $N = \#\Gamma$, and $S_1 = T, S_2, \dots, S_N$ the irreducible representations, where T denotes the trivial one. We only need to see that $e(T) = q - \ell$ and $e(S_i) = -(\ell - 1)$ for $i \geq 2$. Given that, if $R(Z) = aT + \sum_{i\geq 2} a_i S_i$, then $e(F)^{inv} = a$, $e(F) = a + \sum a_i$, and $e(Z) = (q-\ell)a - (\ell-1)\sum a_i = (q-1)a - (\ell-1)(a + \sum a_i)$, as required.

For the trivial representation, it is clear that $e(T) = e(B) = q - \ell$.

Now let S be a (one-dimensional) irreducible representation of Γ. Let $\tilde{\Gamma}$ be the image of $S : \Gamma \to \mathbb{C}^*$, and let $e = \#\tilde{\Gamma}$. Take the e-cover asociated to this group, $\tilde{B} \to B$. Then $\tilde{\Gamma}$ acts on $\tilde{B}$ with quotient B. Clearly, $\tilde{B}$ is a rational curve (the quotient $\Gamma \to \tilde{\Gamma}$ produces a covering map $B_\rho \to \tilde{B}$, and B_ρ is a rational curve by assumption). Then we have a fibration $Y \to \tilde{B} \to B$, where Y is a finite set of e points. Clearly, $R(\tilde{B}) = T + \sum_{p=2}^{e} S_{i_p}$, for some $i_p \in \{2, \dots, N\}$, where $S_{i_2} = S$.

The covering $\tilde{B} \to B$ can be extended to a ramified covering $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$. Hurwitz formula then says that $-2 = e\,(-2) + r$, where r is the degree of the ramification divisor. Then $\varphi^{-1}(q_1, \dots, q_\ell, \infty)$ has $e(\ell+1) - r$ points. Therefore $\tilde{B} = \mathbb{P}^1 - \varphi^{-1}(q_1, \dots, q_\ell, \infty)$ has

$$e(\tilde{B}) = q + 1 - e(\ell+1) + r = q + 1 - e(\ell+1) + (2e-2) = (q-\ell) - (e-1)(\ell-1).$$

The formula in Theorem 2.3.2 says that $e(\tilde{B}) = (q-\ell) + \sum_{p=2}^{e} s_{i_p}$. Therefore $\sum_{p=2}^{e} s_{i_p} = -(e-1)(\ell-1)$.

This happens for all choices of coverings associated to all representations $S_2, \dots, S_N$. Hence $s_i = -(\ell-1)$, for all $i = 2, \dots, N$. $\square$

Let us write γ_z for a small loop around z, so that $H_1(\mathbb{C}^* - \{p_1, \dots, p_n\}) = \langle \gamma_0, \gamma_{p_1}, \dots, \gamma_{p_n} \rangle$. When $n = 2, p_1 = 1, p_2 = -1$, if we consider the $\mathbb{Z}_2$-action given by $\lambda \mapsto \lambda^{-1}$, then $\mathbb{C}^* - \{\pm 1\}/\mathbb{Z}_2 \cong \mathbb{C} - \{\pm 2\}$, so we also write ν_2, ν_{-2} for the free generators of $H_1(\mathbb{C} - \{\pm 2\})$.

Corollary 2.3.6. *Let $\overline{E} \longrightarrow \mathbb{C}^* \setminus \{\pm 1\}$ be a locally trivial fibration with Hodge monodromy representation $R(\overline{E}) \in R(\mathbb{Z}_2)[q]$. Assume that there is a $\mathbb{Z}_2$-action on $\overline{E}$ compatible with the $\mathbb{Z}_2$-action $\lambda \mapsto \lambda^{-1}$ in the base, so that $R(\overline{E}/\mathbb{Z}_2) \in R(\mathbb{Z}_2 \times \mathbb{Z}_2)[q]$. Then we can write*

$$R(\overline{E}/\mathbb{Z}_2) = aT + bS_2 + cS_{-2} + dS_0,$$

where $S_i, i = \pm 2$, denote the irreducible representations characterized by being trivial on the loop ν_i and S_0 is characterized by being trivial on $\nu_2 + \nu_{-2}$. In that case

$$e(\overline{E}) = (q-3)a - 2(b+c+d),$$
$$e(\overline{E}/\mathbb{Z}_2) = (q-2)(a+d) - (b+c).$$

Theorem 2.3.2 also applies to bases of higher dimension. A case that we will use in subsequent chapters is when $B = \mathbb{C}^* \times \mathbb{C}^*$,

Corollary 2.3.7. *Let E be a fibration over $B = \mathbb{C}^* \times \mathbb{C}^*$ with finite and abelian monodromy, such that F is of balanced type. Then Z is also of balanced type and its E-polynomial is given by*

$$e(Z) = (q-1)^2 e(F)^{inv}.$$

Proof. First note that $\pi_1(B) = \mathbb{Z} \times \mathbb{Z}$, so the monodromy Γ, being a quotient $\mathbb{Z} \times \mathbb{Z} \twoheadrightarrow \Gamma$, must be abelian. Moreover, there is some $n > 0$ such that $\mathbb{Z}_n \times \mathbb{Z}_n \twoheadrightarrow \Gamma$, and the covering associated to $\mathbb{Z}_n \times \mathbb{Z}_n$ is of balanced type (is $\mathbb{C}^* \times \mathbb{C}^*$ again). Hence B_ρ is of balanced type, since there are coverings $\mathbb{C}^* \times \mathbb{C}^* \to B_\rho \to \mathbb{C}^* \times \mathbb{C}^*$. Now let S be an irreducible representation of $R(\Gamma)$. If $S = T$, the trivial representation, then $e(T) = e(B) = (q-1)^2$.

If S is a non-trivial representation, then take the covering $\tilde{B} \to B$ associated to S. Again there are coverings $\mathbb{C}^* \times \mathbb{C}^* \to B_\rho \to \tilde{B} \to B = \mathbb{C}^* \times \mathbb{C}^*$. The Hodge monodromy representation associated to $\tilde{B}$ is $T + \sum_{p=2}^{e} S_{i_p}$, where $S_{i_2} = S$. In cohomology $H_c^*(\mathbb{C}^* \times \mathbb{C}^*) \to H_c^*(\tilde{B}) \to H_c^*(B)$ are surjections, but the composition is an isomorphism (multiplication by n). Therefore $e(R(\tilde{B})) = e(\tilde{B}) = (q-1)^2$, and $e(\sum_{p=2}^{e} S_{i_p}) = 0$. This happens for all choices of S, so it must be $e(S) = 0$, for any irreducible non-trivial S. $\square$

There is another property that we shall often use which computes quotients under $\mathbb{Z}_2$-actions. This is particularly useful for our purposes since we will be dealing with $SL(2, \mathbb{C})$. $\mathbb{Z}_2$-actions frequently arise as the interchange of eigenvectors when picking up a basis that diagonalizes a matrix $A \in SL(2, \mathbb{C})$. We will see plenty of examples in the following chapters.

When $\mathbb{Z}_2$ acts on X, we have polynomials $e(X)^+, e(X)^-$, which are the E-polynomials of the invariant and anti-invariant parts of the cohomology of X, respectively. That is

$$e(X)^+ = e(X/\mathbb{Z}_2) \qquad e(X)^- = e(X) - e(X)^+.$$

We have the following proposition for $\mathbb{Z}_2$-actions and fibrations.

Proposition 2.3.8. *Let*

$$\begin{tikzcd} F \arrow[r, "="] \arrow[d] & F \arrow[d] \\ Z \arrow[r] \arrow[d, "\tilde{\pi}"] & Z/\mathbb{Z}_2 \arrow[d, "\pi"] \\ B \arrow[r, "2:1"] & B/\mathbb{Z}_2 \end{tikzcd}$$

be a diagram of fibrations such that B is smooth, irreducible, $\pi, \tilde{\pi}$ are smooth morphisms, $\tilde{\pi}$ is a locally trivial fibration such that the monodromy action of $\pi_1(B)$ is trivial. Then

$$e(Z/\mathbb{Z}_2) = e(F)^+ e(B)^+ + e(F)^- e(B)^-$$

Proof. The monodromy action of $\pi_1(B/\mathbb{Z}_2)$ on the cohomology of the fibre factors through $\mathbb{Z}_2$ and induces a splitting $H_c^*(F) = H_c^*(F) \oplus H_c^*(F)^-$, which, using Poincaré duality, gives a splitting $H^*(F) = H^*(F) \oplus H^*(F)^-$. If we look at the spectral sequences of the fibrations and the restriction map between E_2-terms,

$$H^i(B/\mathbb{Z}_2, H^j(F)) \to H^i(B, H^j(F)) = H^i(B) \otimes H^j(F)$$

the map is compatible with the mixed Hodge structures that the E_2-terms have [1]. It induces an isomorphism

$$H^i(B/\mathbb{Z}_2, H^j(F)) \xrightarrow{\cong} H^i(B, H^j(F))^+ = (H^i(B)^+ \otimes H^j(F)^+) \oplus (H^i(B)^- \otimes H^j(F)^-)$$

which is compatible with mixed Hodge structures. The first spectral sequence abuts to $H^*(Z/\mathbb{Z}_2)$. Since all the differentials go between two groups for which $i + j$ has opposite parity, it follows that any degeneration in the spectral sequence leads to the cancelation of equal terms in the E-polynomial. So

$$e(H^*(Z/\mathbb{Z}_2)) = e(H^*(F)^+)e(H^*(B)^+) + e(H^*(F)^-)e(H^*(B)^-)$$

and the result follows moving again to compactly supported cohomology. □

A similar statement holds when a finite group G acts on a quasiprojective variety Z (see [52]). Then, G also acts on the cohomology $H_c^*(Z)$, compatibly with its mixed Hodge structure. In this situation, $[H_c^*(Z)] \in R(F)$, so an *equivariant Hodge-Deligne polynomial* can be defined as

$$e_G(Z) = \sum_{p,q,k} (-1)^k [H_c^{k,p,q}(Z)] u^p v^q \in R(F)[u, v]$$

Note that the map $\dim R(F) \longrightarrow \mathbb{Z}$ satisfies that $\dim(e_G(Z)) = e(Z)$. For example, take spaces X, X' with a $\mathbb{Z}_2$-action. Then, if we write $e_{\mathbb{Z}_2}(X) = aT + bN, e_{\mathbb{Z}_2}(X') = a'T + b'N$, we have that $e_{\mathbb{Z}_2}(X \times X') = (aa' + bb')T + (ab' + ba')N$, so

$$e((X \times X')/\mathbb{Z}_2) = aa' + bb' = e(X)^+ e(X')^+ + e(X)^- e(X')^-$$

which coincides with Proposition 2.3.8.

For completeness, we finally state the main arithmetic property of E-polynomials. In what follows, let X be an algebraic variety over $\mathbb{C}$.

Definition 2.3.9. *A* spreading out *of X consists of a separated scheme $\mathcal{X}$ over a finitely generated $\mathbb{Z}$-algebra R and an embbeding $\varphi : R \to \mathbb{C}$ such that the extension of scalars verifies that $\mathcal{X}_\varphi \cong X$.*

Definition 2.3.10. *A variety X has* polynomial count *if there is a polynomial $P_X(t) \in \mathbb{Z}[t]$ and a spreading out $\mathcal{X}$ such that for all homomorphisms to a finite field $\phi : R \to \mathbb{F}_q$, the number of points of $\mathcal{X}_\phi$ over $\mathbb{F}_q$ is given by the polynomial P, that is*

$$\#\mathcal{X}_\phi(\mathbb{F}_q) = P_X(q).$$

When the variety is of polynomial type, the following theorem of Katz relates the E-polynomial with its count polynomial, see [37, Appendix 6] for details.

Theorem 2.3.11. *Let X be a variety over $\mathbb{C}$ of polynomial count, with count polynomial $P_X(t)$. Then, the E-polynomial of X is given by*

$$e_X(u, v) = P_X(uv)$$

Katz's theorem was used in [37] to determine the E-polynomials of twisted character varieties, where they showed that these spaces were of polynomial count. They determined their number of points over finite fields using counting formulas that involve character tables of $GL(n, \mathbb{F}_q)$, [32].

2.4 First examples

The following examples are straightforward.

- $e(\mathbb{P}^1(\mathbb{C}) = q + 1$, since

$$h_c^2(\mathbb{P}^1(\mathbb{C})) = h^{2,1,1}(\mathbb{P}^1(\mathbb{C})) = 1$$
$$h_c^0(\mathbb{P}^1(\mathbb{C})) = h_c^{0,0,0}(\mathbb{P}^1(\mathbb{C}) = 1.$$

  and $h_c^{k,i,j}(\mathbb{P}^1(\mathbb{C})) = 0$ otherwise.

- $e(\mathbb{C}) = e(\mathbb{P}^1(\mathbb{C})) - e(\infty) = q$.
- $e(\mathbb{P}^n(\mathbb{C})) = q^n + q^{n-1} + \ldots + 1$. It suffices to decompose $\mathbb{P}^n(\mathbb{C}) = \mathbb{C}^n \sqcup \mathbb{P}^{n-1}(\mathbb{C})$ and proceed inductively.
- $e(\mathbb{C}^n) = q^n$.

Using the locally trivial fibrations in the Zariski topology

$$\begin{aligned} \mathbb{C}^2 \setminus \mathbb{C} \longrightarrow GL(2,\mathbb{C}) &\longrightarrow \mathbb{C}^2 \setminus \{(0,0)\}, \\ \mathbb{C}^* \longrightarrow GL(2,\mathbb{C}) &\longrightarrow PGL(2,\mathbb{C}), \\ \mathbb{C} \longrightarrow SL(2,\mathbb{C}) &\longrightarrow \mathbb{C} \setminus \{(0,0)\}, \end{aligned}$$

and Corollary 2.3.3, we get that

- $e(GL(2,\mathbb{C})) = q(q+1)(q-1)^2$.
- $e(PGL(2,\mathbb{C})) = q^3 - q$.
- $e(SL(2,\mathbb{C})) = q^3 - q$.

We will also need the E-polynomials of the following subgroups of $GL(2,\mathbb{C})$,

$$\begin{aligned} D &:= \{\text{diagonal matrices in } GL(2,\mathbb{C})\} \cong \mathbb{C}^* \times \mathbb{C}^* \\ U &:= \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in GL(2,\mathbb{C}) \right\} \cong \mathbb{C}^* \times \mathbb{C} \end{aligned}$$

It is clear that $e(D) = (q-1)^2$, $e(U) = q^2 - q$. D and U will arise as the stabilizers by the conjugation action on $GL(2,\mathbb{C})$ of a diagonal and a Jordan matrix respectively. We will also need their quotients under the $\mathbb{Z}_2$-action given by left multiplication by $P_0 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, i.e. by interchanging the rows. The action descends to $GL(2,\mathbb{C})/D$. We have

Proposition 2.4.1.

$$\begin{aligned} e(GL(2,\mathbb{C})/D) = q^2 + q \quad & e(GL(2,\mathbb{C})/U) = q^2 - 1 \\ e(GL(2,\mathbb{C})/D)^+ = q^2 \quad & e(GL(2,\mathbb{C})/D)^- = q. \end{aligned}$$

Proof. There is an isomorphism $GL(2,\mathbb{C})/D \cong \mathbb{P}^1 \times \mathbb{P}^1 \setminus \triangle$ given by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto ([a:b],[c:d])$, where $\triangle$ is the diagonal inside $\mathbb{P}^1 \times \mathbb{P}^1$. We obtain $e(GL(2,\mathbb{C})/D) = e(\mathbb{P}^1)^2 - e(\mathbb{P}^1) = q^2 + q$. The $\mathbb{Z}_2$-action interchanges both $\mathbb{P}^1$ factors, precisely $(\mathbb{P}^1 \times \mathbb{P}^1 \setminus \triangle)/\mathbb{Z}_2 \cong \mathbb{P}^2 \setminus C$, where C is a conic. Therefore $e(GL(2,\mathbb{C})/D)^+ = e(\mathbb{P}^2) - e(C) = q^2$. The E-polynomial of $GL(2,\mathbb{C})/U$ can be derived from the locally trivial fibration in the Zariski topology $\mathbb{C}^* \longrightarrow GL(2,\mathbb{C})/U \longrightarrow \mathbb{P}^1$ given by picking the first row of a matrix in $GL(2,\mathbb{C})/D$. We get $e(GL(2,\mathbb{C})/U) = e(\mathbb{C}^*)e(\mathbb{P}^1) = q^2 - 1$. $\square$

Finally, the $\mathbb{Z}_2$-action on $GL(2,\mathbb{C})$ descends to $PGL(2,\mathbb{C})$ since it conmutes with scalar multiplication. We have:

Proposition 2.4.2. $e(PGL(2,\mathbb{C}))^+ = q^3 - q \quad e(PGL(2,\mathbb{C}))^- = 0$

Proof. The $\mathbb{Z}_2$-action on $PGL(2,\mathbb{C})$ comes from the $\mathbb{Z}_2$-action by left-multiplication by P_0 on $GL(2,\mathbb{C})$, which is connected. Therefore the action is homotopically trivial, so the induced action on cohomology is trivial too. This proves the statement. □

We also introduce the following stratification of $SL(2,\mathbb{C})$ by conjugacy types. Consider the following subsets of $\mathrm{SL}(2,\mathbb{C})$:

- $W_0 :=$ conjugacy class of $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. It has $e(W_0) = 1$.
- $W_1 :=$ conjugacy class of $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. It has $e(W_1) = 1$.
- $W_2 :=$ conjugacy class of $J_+ = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. It is $W_2 \cong \mathrm{PGL}(2,\mathbb{C})/U$, with $U = \left\{ \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \mid y \in \mathbb{C} \right\}$. It has $e(W_2) = q^2 - 1$.
- $W_3 :=$ conjugacy class of $J_- = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$. It is $W_3 \cong \mathrm{PGL}(2,\mathbb{C})/U$ and $e(W_3) = q^2 - 1$.
- $W_{4,\lambda} :=$ conjugacy class of $\xi_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, where $\lambda \in \mathbb{C} - \{0, \pm 1\}$. Note that $W_{4,\lambda} = W_{4,\lambda^{-1}}$, since the matrices ξ_λ and $\xi_{\lambda^{-1}}$ are conjugated. We have $W_{4,\lambda} \cong \mathrm{PGL}(2,\mathbb{C})/D$, where $D = \left\{ \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \mid x \in \mathbb{C}^* \right\}$. So $e(W_{4,\lambda}) = q^2 + q$.
- We also need the set $W_4 := \{A \in \mathrm{SL}(2,\mathbb{C}) \mid \mathrm{Tr}\,(A) \neq \pm 2\}$, which is the union of the conjugacy classes $W_{4,\lambda}$, $\lambda \in \mathbb{C} - \{0, \pm 1\}$. This has $e(W_4) = q^3 - 2q^2 - q$, since we can view $e(W_4) = e(GL/D)^+ e(\mathbb{C}^* \setminus \{\pm 1\})^+ + e(GL/D)^- e(\mathbb{C}^* \setminus \{\pm 1\})^- = q^2(q-2) + q(-1)$.

As expected,

$$e(SL(2,\mathbb{C})) = \sum_{i=0}^{4} e(W_i) = q^3 - q.$$

Finally, we also include the following proposition, that will allow us to compute the E-polynomials of several spaces where $GL(2,\mathbb{C})$ acts by conjugation. Let us consider the following space

$$\overline{X} := \left\{ (A_1, \ldots, A_k) \in SL(2,\mathbb{C})^k \mid R(A_1, \ldots, A_k) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \lambda \neq 0, \pm 1 \right\}$$

where $R(A_1, \ldots, A_k)$ is a group relation in terms of $A_1, \ldots, A_k$ (in most of our cases, R will be the single relation defining the fundamental group of a surface of genus g, $R(A_1, \ldots, B_g) = \prod_{i=1}^k [A_i, B_i]$). This gives a locally trivial fibration over $\mathbb{C}^* \setminus \{\pm 1\}$. We can also define

$$\begin{aligned}\tilde{X} := \big\{(A_1, \ldots, A_k, l) \mid R(A_1, \ldots, A_k) = \xi, \operatorname{tr}(\xi) = \lambda + \lambda^{-1},\\ \lambda \neq 0, \pm 1,\ l \text{ eigenspace of } \xi\big\}\end{aligned}$$

and also

$$X := \{(A_1, \ldots, A_k) \mid R(A_1, \ldots, A_k) \sim \xi,\ \operatorname{tr}(\xi) \neq \pm 2\}$$

Note that there is a $\mathbb{Z}_2$-action on $\overline{X}$ given by conjugation by $P_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, covering the $\mathbb{Z}_2$-action on the base given by $\lambda \longrightarrow \lambda^{-1}$. There is a forgetful $2:1$ map from $\tilde{X}$ to X, and also $PGL(2, \mathbb{C}) \times \overline{X}$ is a $D/\mathbb{C}^*$-bundle over $\tilde{X}$, where the map is given by $(P, A_i) \longrightarrow (PA_iP^{-1}, P\begin{pmatrix} 1 \\ 0 \end{pmatrix})$.

When $R(\overline{X}/\mathbb{Z}_2) \in R(\mathbb{Z}_2 \times \mathbb{Z}_2)[q]$, we can compute the E-polynomial of X in terms of $R(\overline{X}/\mathbb{Z}_2)$.

Proposition 2.4.3. *Suppose that we have a diagram*

$$\begin{tikzcd} \mathbb{C}^* \arrow[r] \arrow[d] & PGL(2,\mathbb{C}) \times \overline{X} \arrow[r] \arrow[d] & \tilde{X} \arrow[d, "2:1"] \\ \mathbb{C}^* \arrow[r] & (PGL(2,\mathbb{C}) \times \overline{X})/\mathbb{Z}_2 \arrow[r] & X \end{tikzcd}$$

where $\overline{X}$ is a locally fibration over $\mathbb{C}^* \setminus \{\pm 1\}$, compatible with the $\mathbb{Z}_2$-action $\lambda \mapsto \lambda^{-1}$ in the base, and such that $PGL(2, \mathbb{C})$ acts by conjugation on $\overline{X}$, with fibre $D/\mathbb{C}^* \cong \mathbb{C}^*$ given by $\mu \sim \begin{pmatrix} \mu & 0 \\ 0 & 1 \end{pmatrix}$. Assume that the $\mathbb{Z}_2$-action on the fibre $\mathbb{C}^*$ takes $\mu \mapsto \mu^{-1}$. Then

$$e(X) = q(q^2 - 2q - 1)a - q(q+1)(b+c) - 2qd,$$

where $R(\overline{X}/\mathbb{Z}_2) = aT + bS_2 + cS_{-2} + dS_0$.

Proof. First of all, using Corollary 2.3.3, we get that

$$e(\tilde{X}) = \frac{e(PGL(2, \mathbb{C})e(\overline{X})}{e(\mathbb{C}^*)} = q(q+1)e(\overline{X})$$

so that

$$\begin{aligned} e((PGL(2,\mathbb{C}) \times \overline{X})/\mathbb{Z}_2) &= e(\tilde{X})^+ e(\mathbb{C}^*)^+ + e(\tilde{X})^- e(\mathbb{C}^*)^- \\ &= e(X)q - (e(\tilde{X}) - e(X)) \\ &= (q+1)e(X) - q(q+1)e(\overline{X}). \end{aligned}$$

On the other hand, using Proposition 2.3.8,

$$\begin{aligned} e((PGL(2,\mathbb{C})\times\overline{X})/\mathbb{Z}_2) &= e(PGL(2,\mathbb{C})\times\overline{X})^+ \\ &= e(PGL(2,\mathbb{C}))^+e(\overline{X})^+ + e(PGL(2,\mathbb{C}))^-e(\overline{X})^- \\ &= e(PGL(2,\mathbb{C}))e(\overline{X}/\mathbb{Z}_2), \end{aligned}$$

which gives us the equality

$$q(q^2-1)e(\overline{X}/\mathbb{Z}_2) = (q+1)e(X) - q(q+1)e(\overline{X})$$

from where we deduce

$$e(X) = (q^2-q)e(\overline{X}/\mathbb{Z}_2) + qe(\overline{X}). \tag{2.6}$$

Finally, using the Hodge monodromy representations and Corollary 2.3.6

$$\begin{aligned} e(X) &= (q^2-q)e(\overline{X})^+ + qe(\overline{X}) \\ &= q(q-1)((q-2)a - (b+c+d)) + q((q-3)(a+d) - 2(b+c)) \\ &= q(q^2-2q-1)a - q(q+1)(b+c) - 2qd. \end{aligned}$$

□

Chapter 3

Basic pieces. Genus 1 and 2 computations

3.1 Basic pieces: $SL(2,\mathbb{C})$-character varieties for $g = 1, 2$

In this chapter, we introduce what we will call basic pieces: they correspond to the representation spaces of the fundamental group of a once-punctured surface of genus 1 and 2 into $SL(2,\mathbb{C})$, where the monodromy around the puncture is fixed. They are the building blocks for the computation of the E-polynomials associated to curves of genus $g \geq 3$, that will be the core of Chapters 4 and 5.

If we write $c \in \pi_1(X)$ for a generator for the loop around the puncture, the fundamental group of the punctured surface X has the following presentation: $\pi_1(X) = \langle a, b, c \mid [a,b]c = e\rangle$. We are looking for representations of this fundamental group that send c to a fixed element $C \in SL(2,\mathbb{C})$, belonging to a certain conjugacy class $\mathcal{C}$. Explicitly, these spaces are defined as follows:

$$\begin{aligned}
\overline{X}_0 &:= X_0 := \{(A,B) \in SL(2,\mathbb{C})^2 \mid [A,B] = \mathrm{Id}\},\\
\overline{X}_1 &:= X_1 := \{(A,B) \in SL(2,\mathbb{C})^2 \mid [A,B] = -\mathrm{Id}\},\\
\overline{X}_2 &:= \{(A,B) \in SL(2,\mathbb{C})^2 \mid [A,B] = J_+\},\\
X_2 &:= \{(A,B) \in SL(2,\mathbb{C})^2 \mid [A,B] \sim J_+\},\\
\overline{X}_3 &:= \{(A,B) \in SL(2,\mathbb{C})^2 \mid [A,B] = J_-\},\\
X_3 &:= \{(A,B) \in SL(2,\mathbb{C})^2 \mid [A,B] \sim J_-\},\\
\overline{X}_{4,\lambda} &:= \left\{(A,B) \in SL(2,\mathbb{C})^2 \mid [A,B] = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}\right\}, \lambda \neq 0, \pm 1,\\
\overline{X}_4 &:= \left\{(A,B,\lambda) \in SL(2,\mathbb{C})^2 \times \mathbb{C}^* \setminus \{\pm 1\} \mid [A,B] = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}\right\},\\
X_4 &:= \left\{(A,B,\lambda) \in SL(2,\mathbb{C})^2 \times \mathbb{C}^* \setminus \{\pm 1\} \mid [A,B] \sim \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}\right\}.
\end{aligned}$$

Note that $SL(2,\mathbb{C})^2 = \sqcup_{i=0}^4 X_i$. In other words, we can use the stratification of $SL(2,\mathbb{C})^2$ given by the map

$$\begin{aligned} \tilde{f} : SL(2,\mathbb{C})^2 &\longrightarrow SL(2,\mathbb{C}) \\ (A,B) &\mapsto [A,B] = ABA^{-1}B^{-1} \end{aligned} \tag{3.1}$$

so that $X_i = f^{-1}(W_i)$, $i = 0,\ldots,4$ and $SL(2,\mathbb{C})^2 = f^{-1}(SL(2,\mathbb{C})) = \sqcup_{i=0}^4 X_i$. Every X_i is an affine algebraic set in $SL(2,\mathbb{C})^2$, defined by a single equation. Using a convenient stratification, one can describe the action of $\mathrm{PGL}(2,\mathbb{C})$ by conjugation and at the same time divide each representation space in smaller locally closed subspaces whose E-polynomials are easier to compute. These calculations were performed in [53]. The E-polynomials of a once-punctured surface are

$$\begin{aligned}
e(\overline{X}_0) &= q^4 + 4q^3 - q^2 - 4q, \\
e(\overline{X}_1) &= q^3 - q, \\
e(\overline{X}_2) &= q^3 - 2q^2 - 3q, \\
e(X_2) &= q^5 - 2q^4 - 4q^3 + 2q^2 + 3q, \\
e(\overline{X}_3) &= q^3 + 3q^2, \\
e(X_3) &= q^5 + 3q^4 - q^3 - 3q^2, \\
e(\overline{X}_{4,\lambda}) &= q^3 + 3q^2 - 3q - 1, \\
e(\overline{X}_4) &= q^4 - 3q^3 - 6q^2 + 5q + 3, \\
e(X_4) &= q^6 - 2q^5 - 4q^4 + 3q^2 + 2q.
\end{aligned}$$

Besides, the E-polynomials of the associated character varieties $\mathcal{M}_{\mathcal{C}} := X_i//SL(2,\mathbb{C})$ are:

$$\begin{aligned}
e(\mathcal{M}^{g=1}_{\mathrm{Id}}) &= q^2 + 1, \\
e(\mathcal{M}^{g=1}_{-\mathrm{Id}}) &= 1, \\
e(\mathcal{M}^{g=1}_{J_+}) &= q^2 - 2q - 3, \\
e(\mathcal{M}^{g=1}_{J_-}) &= q^2 + 3q, \\
e(\mathcal{M}^{g=1}_{\xi_\lambda}) &= q^2 + 4q + 1.
\end{aligned}$$

Note that in the cases where the quotient is geometric, it suffices to divide by $e(\mathrm{Stab}\,(C))$ (it is not the case for $\mathcal{M}^{g=1}_{\mathrm{Id}}$, for example). A crucial result for our purposes is also given in [53]: the analysis of the behaviour of the parabolic character variety $\mathcal{M}^{g=1}_{\xi_\lambda}$ when the parameter λ changes. The map $(A,B,\lambda) \in \overline{X}_4 \mapsto \lambda$ fibres this space over $\mathbb{C} \setminus \{0,\pm 1\}$ with fibre $\overline{X}_{4,\lambda}$, but the fibration is only analytically trivial and there is monodromy. The Hodge monodromy representation, introduced in Chapter 2, encodes this variation and captures the E-polynomial of the invariant and non-invariant part of the cohomology of the fibre under the monodromy of the fibration.

Let us write $H_1(\mathbb{C} \setminus \{0, \pm 1\}) = \langle \gamma_0, \gamma_1, \gamma_{-1} \rangle$. It is proved in [53] that the monodromy around the origin is an involution and that it is trivial around γ_{-1}, γ_1, so that $R(\overline{X}_4) \in R(\mathbb{Z}_2)[q]$. Specifically,

$$R(\overline{X}_4) = (q^3 - 1)T + (3q^2 - 3q)N \in R(\mathbb{Z}_2)[q], \tag{3.2}$$

where T, N are the trivial and non-trivial representations respectively. There is a $\mathbb{Z}_2$-action given by conjugation by P_0 on $\overline{X}_4$, which corresponds to the permutation of the eigenvectors. The quotient space can be seen as a locally trivial fibration over $\mathbb{C} \setminus \{\pm 2\}$ using the map $(A, B, \lambda) \in \overline{X}_4/\mathbb{Z}_2 \mapsto s := \lambda + \lambda^{-1}$. Let us write $H_1(\mathbb{C} \setminus \{\pm 2\}) = \langle \nu_2, \nu_{-2} \rangle$ and $\gamma_0 := \nu_2 + \nu_{-2}$. Then the fibration has Hodge monodromy representation

$$R(\overline{X}_4/\mathbb{Z}_2) = q^3 T - 3qS_2 + 3q^2 S_{-2} - S_0 \in R(\mathbb{Z}_2 \times \mathbb{Z}_2)[q], \tag{3.3}$$

where S_0, S_{-2}, S_2 is the representation characterized by being trivial on $\gamma_0, \nu_{-2}, \nu_2$ respectively. As we pointed out in Corollary 2.3.6, the E-polynomials of the fibre and the total space of the fibration can be recovered from the Hodge Monodromy Representation, as well as the total polynomial of X_4. In this particular case, if we write $R(\overline{X}_4/\mathbb{Z}_2) = aT + bS_2 + cS_{-2} + dS_0$,

$$\begin{aligned} e(\overline{X}_{4,\lambda}) &= a + b + c + d \\ e(\overline{X}_4/\mathbb{Z}_2) &= (q-2)a - (b + c + d) \\ e(\overline{X}_4) &= (q-3)(a+d) - 2(b+c). \end{aligned}$$

For the genus 2 case,

$$\mathcal{M}_{\mathcal{C}}^{g=2} := \{(A, B, C, D) \mid [A, B][C, D] = C\} // \operatorname{Stab}(C),$$

a stratification of each of the representation spaces was made using a slight variation of the trace map of the commutators, in the spirit of (3.1). More concretely,

$$\begin{aligned} \tilde{f} : SL(2, \mathbb{C})^4 &\longrightarrow \mathbb{C}^2 \\ (A, B) &\mapsto (\operatorname{tr}[D, C], \operatorname{tr}[A, B]) \end{aligned} \tag{3.4}$$

This map allowed to study the representation space as a fibration over $\mathbb{C}^2$, where the fibres when any trace equals ± 2 are of different type and a separate treatment in these cases is needed. To obtain the E-polynomials of the associated moduli spaces, it suffices again to divide by $e(\mathrm{Stab}\,(C))$ in the cases when the quotient is geometric and to study the set of reducible orbits in the case where a GIT quotient is needed.

In this chapter, we use the same tools to compute the E-polynomials of the character variety of the fundamental group of some non-orientable surfaces: the Klein bottle K and the connected sum of three projective planes, $\#^3 P^2$. They have non-orientable genus 1 and 2 respectively. These character varieties are of interest in non-abelian Hodge theory: they have recently been studied in [49], where a Donaldson-Corlette correspondence has been stablished.

For the Klein bottle case, we stratify $SL(2,\mathbb{C})^2$ and calculate slices for the action of $\mathrm{PGL}(2,\mathbb{C})$, to obtain explicit descriptions of the moduli spaces. For the genus 2 case of the connected sum of three projective planes, we fibre the character varieties using a trace map and compute the GIT quotient in each case. Unlike the orientable case, there are reducible orbits in every stratum and a detailed study of each case is needed. We also explore the relation with the orientable case given by the map induced by the oriented double covers of these spaces.

The main theorem of this chapter is the computation of the E-polynomials of these spaces.

Theorem 3.1.1. *Let K be the Klein bottle. The E-polynomials of the* $\mathrm{SL}(2,\mathbb{C})$*-character varieties $\mathcal{M}_\xi(K)$ are*

$$\begin{aligned} e(\mathcal{M}_{\mathrm{Id}})(K) &= 3q-2, \\ e(\mathcal{M}_{-\mathrm{Id}})(K) &= q-1, \\ e(\mathcal{M}_{J_+})(K) &= q^2+2q-7, \\ e(\mathcal{M}_{J_-})(K) &= q^2+3q, \\ e(\mathcal{M}_{\xi_\lambda})(K) &= q^2+2q+1. \end{aligned}$$

Theorem 3.1.2. *Let Σ be the connected sum of three projective planes. The E-polynomials of its associated* $\mathrm{SL}(2,\mathbb{C})$*-character varieties are*

$$\begin{aligned} e(\mathcal{M}_{\mathrm{Id}})(\Sigma) &= q^3-6q-1, \\ e(\mathcal{M}_{-\mathrm{Id}})(\Sigma) &= 2q^3+7q^2-1, \\ e(\mathcal{M}_{J_+})(\Sigma) &= q^5+5q^3+12q^2-8q+26, \\ e(\mathcal{M}_{J_-})(\Sigma) &= q^5-5q^3-12q^2, \\ e(\mathcal{M}_{\xi_\lambda})(\Sigma) &= q^5+q^4+2q^3+8q^2-27q+23. \end{aligned}$$

3.2 Stratification of $SL(2,\mathbb{C})^2$

Our first goal is to stratify $SL(2,\mathbb{C})^2$ according to the map

$$\begin{aligned}\tilde{f}: SL(2,\mathbb{C})^2 &\longrightarrow SL(2,\mathbb{C})\\ (A,B) &\mapsto ABAB^{-1}\end{aligned}$$

by taking inverse images of the stratification of $SL(2,\mathbb{C}) = \sqcup_{i=0}^4 W_i$ defined in Section 2.4. Note that if we consider the action by conjugation of $\mathrm{PGL}(2,\mathbb{C})$ on $SL(2,\mathbb{C})$, the map $\tilde{f}$ is $\mathrm{PGL}(2,\mathbb{C})$-equivariant. Therefore, writing $Y_i := \tilde{f}^{-1}(W_i)$,

$$\begin{aligned}SL(2,\mathbb{C})^2 &= \tilde{f}^{-1}(W_0) \sqcup \tilde{f}^{-1}(W_1) \sqcup \tilde{f}^{-1}(W_2) \sqcup \tilde{f}^{-1}(W_3) \sqcup \tilde{f}^{-1}(W_4)\\ &= Y_0 \sqcup Y_1 \sqcup Y_2 \sqcup Y_3 \sqcup Y_4.\end{aligned}$$

We will frequently compute slices for the $\mathrm{PGL}(2,\mathbb{C})$-action that will be useful for the description of the moduli spaces and for the computation of the E-polynomials. We proceed with the computation of each Y_i.

Y_0.

By definition,

$$Y_0 = \{(A,B) \in SL(2,\mathbb{C})^2 \mid ABAB^{-1} = \mathrm{Id}\}.$$

We subdivide Y_0 according to the different cases for (A,B):

- $Y_0^1 := \{(A,B) \in Y_0 \mid A = \pm\,\mathrm{Id}\}$. In this case, the equation is trivial and B can be any matrix belonging to $SL(2,\mathbb{C})$. Therefore

  $$Y_0^1 \cong \{\pm\,\mathrm{Id}\} \times SL(2,\mathbb{C}),$$

  so we get

  $$e(Y_0^1) = 2(q^3 - q).$$

  Notice that also if $B = \pm\,\mathrm{Id}$, then $A^2 = \mathrm{Id}$, whose only solutions in $SL(2,\mathbb{C})$ are $A = \pm\,\mathrm{Id}$. So the cases where A or B are equal to $\pm\,\mathrm{Id}$ are contained in this stratum. Therefore,we can assume in the following cases that $A, B \neq \pm\,\mathrm{Id}$.

- $Y_0^2 = \{(A,B) \in Y_0 \mid A \text{ is diagonalizable, } A \neq \pm\,\mathrm{Id}\}$. In this case, we can fix a basis such that

  $$A = \begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix},\ \lambda \neq 0, \pm 1.$$

If we write $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then the relation $ABA = B$ gives us the system of equations

$$\begin{cases} \lambda^2 a = a \\ \frac{d}{\lambda^2} = d \end{cases}$$

which forces $a = d = 0$. We obtain that, with respect to this basis,

$$B = \begin{pmatrix} 0 & b \\ -\frac{1}{b} & 0 \end{pmatrix}.$$

Rescaling the eigenvectors (i.e. conjugating by a diagonal matrix, which leaves A invariant), we can also assume that $b = i$. Notice that P_0 takes λ to λ^{-1} and leaves B invariant, so $\lambda \sim \lambda^{-1}$. Therefore

$$Y_0^2 \cong \mathrm{PGL}(2, \mathbb{C}) \times (\mathbb{C}^* \setminus \{\pm 1\}) / \mathbb{Z}_2.$$

Since $(\mathbb{C}^* \setminus \{\pm 1\}) / \mathbb{Z}_2 \cong \mathbb{C} \setminus \{\pm 2\}$ (via the map $\lambda \mapsto s = \lambda + \lambda^{-1}$), we obtain (using Proposition 2.3.8 and the fact that $e(PGL(2, \mathbb{C}))^- = 0$)

$$e(Y_0^2) = (q^3 - q)(q - 2).$$

- $Y_0^3 = \{(A, B) \in Y_0 \mid A \sim J_+\}$. As we did before, we can fix a basis such that $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then the relation $ABA = B$ produces the equations

$$\begin{cases} a + c = a \\ c + d = d \\ a + b + c + d = b, \end{cases}$$

which gives us $c = a + d = 0$. Using that $\det(B) = 1$, then $ad = -a^2 = 1$, so B takes the form

$$\begin{pmatrix} \pm i & b \\ 0 & \mp i \end{pmatrix}.$$

To get a slice for the $\mathrm{PGL}(2, \mathbb{C})$-action, we can conjugate by a matrix of the form $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, which leaves A invariant, and assume that $b = 0$. We get that this stratum has two components ($a = \pm i$), so

$$Y_0^3 \cong \{\pm i\} \times \mathrm{PGL}(2, \mathbb{C}),$$

and we conclude that

$$e(Y_0^3) = 2(q^3 - q).$$

- $Y_0^4 = \{(A, B) \in Y_0 \mid A \sim J_-\}$. Similar calculations to the ones made for Y_0^3 yield

$$e(Y_0^4) = 2(q^3 - q).$$

Adding up, we obtain

$$\begin{aligned} e(Y_0) &= e(Y_0^1) + e(Y_0^2) + e(Y_0^3) + e(Y_0^4) \\ &= 2(q^3-q) + (q^3-q)(q-2) + 2(q^3-q) + 2(q^3-q) \\ &= (q^3-q)(q+4). \end{aligned}$$

Y_1.

By definition, we have that $Y_1 = \{(A,B) \in SL(2,\mathbb{C})^2 \mid ABA = -B\}$. Note that we have the identity $\operatorname{tr} A = \operatorname{tr}(BAB^{-1}) = \operatorname{tr}(-A^{-1}) = -\operatorname{tr} A$, so $\operatorname{tr} A = 0$. With respect to a certain basis, A takes the diagonal form

$$A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

If we write $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the equation $ABA = -B$ yields that $b = c = 0$, so

$$B = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix},$$

with $x \in \mathbb{C}^*$. The set of diagonal matrices $D \subset GL(2,\mathbb{C})$ acts trivially on these pairs, so

$$Y_1 \cong \mathbb{C}^* \times \mathrm{PGL}(2,\mathbb{C})/D$$

and therefore

$$e(Y_1) = (q-1)(q^2+q) = q^3 - q.$$

Y_2.

In this case, we have to distinguish between

$$\overline{Y}_2 := \{(A,B) \in SL(2,\mathbb{C})^2 \mid ABAB^{-1} = J_+\}$$

and

$$Y_2 = \{(A,B) \in SL(2,\mathbb{C})^2 \mid ABAB^{-1} \sim J_+\}.$$

We note, as we did in the previous case, that the following identity for the traces holds:

$$\operatorname{tr} A = \operatorname{tr} BAB^{-1} = \operatorname{tr} A^{-1}J_+.$$

If we write $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then we obtain that $a + d = a + d - c$, so $c = 0$ and

$$A = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}.$$

To get slices for the conjugation action, let us deal first with $\overline{Y}_2$. If $B = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$, the relation

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} x & y \\ z & t \end{pmatrix} \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & y \\ z & t \end{pmatrix}$$

gives us the set of equations (adding the equation $\det B = 1$)

$$\begin{cases} xt - zy = 1 \\ x + z = a(ax + bz) \\ y + t = b(ax + bz) + y + \frac{b}{a}t \\ t = \frac{bz}{a} + \frac{t}{a^2}. \end{cases}$$

Notice the following: conjugating by an element $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, which belongs to $\mathrm{Stab}\,(J_+)$, takes A to

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \longrightarrow \begin{pmatrix} a & b - x(a - a^{-1}) \\ 0 & a^{-1} \end{pmatrix}.$$

So we subdivide again the stratum into two parts: according to whether $(a - a^{-1})$ is zero or not, i.e., $a^2 \neq 1$ and $a^2 = 1$.

- $\overline{Y}_2^1 := \{(A, B) \in \overline{Y}_2 \mid a^2 \neq 1\}$. In this case, A is diagonalizable and, by the previous calculation, we can assume that $b = 0$. We get the equations

$$\begin{cases} xt - zy = 1 \\ x + z = a^2 x \\ y + t = y \\ t = \frac{t}{a^2}, \end{cases}$$

  from which we deduce that $t = 0$, $z = (a^2 - 1)x$ and therefore $y = \frac{-1}{(a^2-1)x}$, $x \in \mathbb{C}^*$. So

$$A = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \quad B = \begin{pmatrix} x & \frac{-1}{(a^2-1)x} \\ (a^2 - 1)x & 0 \end{pmatrix}.$$

  We get a slice S for the conjugation action isomorphic to $\mathbb{C}^* \times \mathbb{C}^* \setminus \{\pm 1\}$. So

$$Y_2^1 \cong S \times \mathrm{PGL}(2, \mathbb{C}) \cong \mathbb{C}^* \times \mathbb{C}^* \setminus \{\pm 1\} \times \mathrm{PGL}(2, \mathbb{C}),$$

  and therefore

$$e(Y_2^1) = (q - 1)(q - 3)(q^3 - q).$$

- $\overline{Y}_2^2 := \{(A, B) \in \overline{Y}_2 \mid a^2 = 1\}$. We deal with the case $a = 1$, the other case being similar. If $a = 1$, the equations are

$$\begin{cases} xt - zy = 1 & (3.5) \\ x + z = x + bz & (3.6) \\ y + t = b(x + bz) + y + bt & (3.7) \\ t = bz + t & (3.8) \end{cases}$$

We deduce from (3.6) and (3.8) that $z = 0$, so from (3.5), $t = x^{-1}$ and from (3.7) $t = b(x + x^{-1})$. If $x = \pm i$, there is no solution. If $x \neq \pm i$, then $b = \frac{1}{x^2+1}$. So finally, A and B take the form

$$A = \begin{pmatrix} 1 & \frac{1}{x^2+1} \\ 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix}$$

Now, if $x \neq \pm 1$, conjugating by an element of $\mathrm{Stab}\,(J_+)$, we can assume that $y = 0$. If $x = \pm 1$, every element of $\mathrm{Stab}\,(J_+)$ also stabilizes both A and B. Dividing in two cases, and doubling the contribution to account for the case $a = -1$, we obtain

$$\overline{Y}_2^2 \cong \bigsqcup_{j=1}^{2} \big((\mathbb{C} \setminus \{0, \pm 1, \pm i\} \times \mathrm{Stab}\,(J_+)) \sqcup \mathbb{C} \sqcup \mathbb{C} \big),$$

so we get

$$e(\overline{Y}_2^2) = 2(q(q-5) + 2q) = 2q^2 - 6q.$$

Moreover

$$Y_2^2 \cong \bigsqcup_{j=1}^{2} \big((\mathbb{C} \setminus \{0, \pm 1, \pm i\} \times \mathrm{PGL}(2, \mathbb{C})) \sqcup (\mathbb{C} \times \mathrm{PGL}(2, \mathbb{C})/U) \sqcup (\mathbb{C} \times \mathrm{PGL}(2, \mathbb{C})/U) \big)$$

and

$$\begin{aligned} e(Y_2^2) &= 2((q^3 - q)(q-5) + 2q(q^2 - 1)) \\ &= 2(q^3 - q)(q - 3). \end{aligned}$$

Finally

$$\begin{aligned} e(Y_2) &= e(Y_2^1) + e(Y_2^2) \\ &= (q^3 - q)((q-1)(q-3) + 2(q-3)) \\ &= (q^3 - q)(q+1)(q-3). \end{aligned}$$

Y_3.

Let $\overline{Y}_3 = \{(A, B) \in SL(2, \mathbb{C})^2 \mid ABAB^{-1} = J_-\}$, and $Y_3 = \{(A, B) \in SL(2, \mathbb{C})^2 \mid ABAB^{-1} \sim J_-\}$. The equation $\operatorname{tr} A = \operatorname{tr} A^{-1} J_-$ leads to the equation $c = -2(a + d)$, so

$$A = \begin{pmatrix} a & b \\ -2(a+d) & d \end{pmatrix}.$$

Y_3^1.

Let us deal first with the case $a+d=0$. Then $a+d=0$ and $ad=1$ force that $a=\pm i$, and

$$A=\begin{pmatrix}\pm i & b\\ 0 & \mp i\end{pmatrix}.$$

The equations $ABA=J_-B, \det(B)=1$ give us the set of equations

$$\begin{cases}\pm ibz-x=z-x & (3.9)\\ \pm ixb+b^2z+y\mp ibt=t-y & (3.10)\\ z=-z & (3.11)\\ \mp zbi-t=-t. & (3.12)\end{cases}$$

We deduce immediately from equation (3.11) that $z=0$, so equations (3.9) and (3.12) are trivial. From equation (3.10) and $xt=1$ we deduce that $y=\frac{1\mp ix^2b\pm ib}{2x}$. Therefore

$$A=\begin{pmatrix}\pm i & b\\ 0 & \mp i\end{pmatrix},\quad B=\begin{pmatrix}x & \frac{1\mp ix^2b\pm ib}{2x}\\ 0 & x^{-1}\end{pmatrix}.$$

To obtain a slice for the action, we can assume that $b=0$ conjugating by an element in $\mathrm{Stab}\,(J_-)$. The slice is isomorphic then to 2 copies of $\mathbb{C}^*$, so

$$e(\overline{Y}_3^1)=2q(q-1)\quad e(Y_3^1)=2(q-1)(q^3-q).$$

Y_3^2.

We focus now on the case $a+d\neq 0$. Conjugating by an element of $\mathrm{Stab}\,(J_-)$, we can also assume that $d=0$. So A takes the form

$$A=\begin{pmatrix}a & \frac{1}{2a}\\ -2a & 0\end{pmatrix}.$$

Now if $B=\begin{pmatrix}x & y\\ z & t\end{pmatrix}$, the equation $ABA=J_-B$ leads us to the equations

$$\begin{cases}a^2x+\frac{z}{2}-2a^2y-t=z-x\\ \frac{x}{2}+\frac{z}{4a^2}=t-y\\ -2a^2x+4a^2y=-z\\ -x=-t.\end{cases}$$

So we get that $x=t$, and an expression of z in terms of x,y. It turns out, after substituting in the other two equations, that it is the only equation for B. Therefore

$$B=\begin{pmatrix}x & y\\ 2a^2(x-2y) & x\end{pmatrix},$$

where the equation $\det B = 1$ becomes

$$x^2 - 2a^2yx + 4a^2y^2 = 1. \tag{3.13}$$

The equation is a conic for fixed a, so we get a conic bundle if we choose $a \in \mathbb{C}^*$ to be the base. We have therefore constructed a slice for the conjugation action in $\overline{Y}_3^2$ and it remains to compute its polynomial.

The discriminant of the conic (3.13) is $D = 4a^2(a^2-4)$, so if $a = \pm 2$, we get two parallel lines for each of such values, which gives us a contribution of $4q$. If $a \neq \pm 2$, let E be the conic bundle obtained by adding on the points at infinity to each of the above conics, i.e.

$$x^2 - 2a^2yx + 4a^2y^2 = z^2.$$

This gives us a conic bundle $\mathbb{P}^1 \longrightarrow E \longrightarrow \mathbb{C}^* \setminus \{\pm 2\}$ with E-polynomial

$$e(E) = e(\mathbb{P}^1)e(\mathbb{C}^* \setminus \{\pm 2\}) = (q+1)(q-3).$$

From this conic bundle, we have to remove the points at infinity. Writing $t = \frac{x-a^2y}{ay}$, we have the equation

$$t^2 = a^2 - 4.$$

This conic has two points at infinity and we have to remove the points $(a,t) = (0, \pm 2i), (\pm 2, 0)$, so its E-polynomial is $q-5$. Since $\overline{Y}_3^2 \cong \mathrm{Stab}\,(J_-) \times S$, we obtain

$$e(\overline{Y}_3^2) = q((q+1)(q-3) - (q-5) + 4q) = q(q^2+q+2)$$

and therefore

$$e(Y_3^2) = (q^3-q)(q^2+q+2).$$

Finally

$$\begin{aligned} e(Y_3) &= e(Y_3^1) + e(Y_3^2) \\ &= (q^3-q)(2(q-1) + (q^2+q+2)) \\ &= (q^3-q)(q^2+3q). \end{aligned}$$

Y_4.

We consider:

$$\overline{Y}_4 := \left\{ (A, B, \lambda) \in SL(2,\mathbb{C})^2 \times (\mathbb{C}^* \setminus \{\pm 1\}) \mid ABAB^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \right\}.$$

For fixed $\lambda \neq 0, \pm 1$, we define also

$$\overline{Y}_{4,\lambda} := \left\{ (A, B) \in SL(2,\mathbb{C})^2 \mid ABAB^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \right\}$$

Moreover

$$Y_4 = \left\{ (A,B) \in SL(2,\mathbb{C})^2 \mid ABAB^{-1} \sim \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \text{ for some } \lambda \neq 0, \pm 1 \right\}$$

There is a $\mathbb{Z}_2$-action on $\overline{Y}_4$ given by the permutation of the eigenvalues. If we write $P_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, P_0 acts on $\overline{Y}_4$ taking (A, B, λ) to $(P_0^{-1}AP_0, P_0^{-1}BP_0, \lambda^{-1})$. Write

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B = \begin{pmatrix} x & y \\ z & t \end{pmatrix}.$$

Now, the equation $\operatorname{tr} A = \operatorname{tr} BAB^{-1} = \operatorname{tr}\left(A^{-1}\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)\right)$ implies that $d = \frac{a}{\lambda}$, so

$$A = \begin{pmatrix} a & b \\ c & \frac{a}{\lambda} \end{pmatrix}.$$

We also have the equations $AB = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} BA^{-1}$, and $\det A = \det B = 1$. These lead us to the equations

$$\begin{cases} bz + \lambda yc = 0 & (3.14) \\ (\lambda - 1)ay - b(\lambda x + t) = 0 & (3.15) \\ (\lambda - 1)az + \lambda c(\lambda x + t) = 0 & (3.16) \\ xt - yz = 1 & (3.17) \\ \dfrac{a^2}{\lambda} - bc = 1. & (3.18) \end{cases}$$

We stratify Y_4 in different strata Y_4^i and construct a slice S^i for the $\mathrm{PGL}(2,\mathbb{C})$-action in each one. Since the stabilizer of $\xi = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$ in $\mathrm{PGL}(2,\mathbb{C})$ is isomorphic to $\mathbb{C}^*$, writing S_λ^i for $S^i \cap \overline{Y}_{4,\lambda}$,

$$\overline{Y}_{4,\lambda}^i \cong \mathbb{C}^* \times S_\lambda^i \qquad \overline{Y}_4^i \cong \mathbb{C}^* \times S^i.$$

If the slices are invariant under the action, then

$$Y_4^i \cong (\mathrm{PGL}(2,\mathbb{C}) \times S^i)/\mathbb{Z}_2$$

and, since $e(\mathrm{PGL}(2,\mathbb{C}))^- = 0$, we get that $e(Y_4^i) = e(S^i/\mathbb{Z}_2)e(\mathrm{PGL}(2,\mathbb{C}))$.

$Y_4^1 := \{(A,B) \in Y_4 \mid bc = 0\}$

We stratify again according to three possible cases, $Y_4^{1,a}, Y_4^{1,b}$ and $Y_4^{1,c}$, depending on the different options for b and c.

- $Y_4^{1,a} := \{(A,B) \in Y_4 \mid b = 0, c \neq 0\}$. We see, from equations (3.14)-(3.18), that if $y \neq 0$, then we would get $a = c = 0$ and no solutions. So $y = 0$, and equations (3.14)-(3.18) in this case are:

$$\begin{cases} (\lambda - 1)az + \lambda c(\lambda x + t) = 0 \\ xt = 1 \\ a^2 = \lambda. \end{cases}$$

So A and B are of the form

$$A = \begin{pmatrix} a & 0 \\ c & \frac{a}{\lambda} \end{pmatrix}, \quad B = \begin{pmatrix} x & 0 \\ \frac{\lambda c(\lambda x + x^{-1})}{(1-\lambda)a} & x^{-1} \end{pmatrix},$$

where $\frac{a^2}{\lambda} = 1$. Since $c \neq 0$, we can conjugate by an element of $\operatorname{Stab}(\xi_\lambda)$ and assume that $c = 1$ for our slice. We get that $S_{4,\lambda}^{1,a} \cong \mathbb{C}^* \times \{\pm\sqrt{\lambda}\}$ and $S_4^{1,a} \cong \mathbb{C}^* \times \mathbb{C} \backslash \{0, \pm i, \pm 1\}$. Therefore

$$\begin{aligned} e(\overline{Y}_{4,\lambda}^{1,a}) &= 2(q-1)^2 \\ e(\overline{Y}_4^{1,a}) &= (q-5)(q-1)^2. \end{aligned}$$

- $Y_4^{1,b} := \{(A,B) \in Y_4 \mid b \neq 0, c = 0\}$. Again from equations (3.14) and (3.16), we see that if $z \neq 0$, then we would have $b = a = 0$, a contradiction. Therefore, necessarily $z = 0$ and we get the equations

$$\begin{cases} (\lambda - 1)ay - b(\lambda x + t) = 0 \\ xt = 1 \\ a^2 = \lambda. \end{cases}$$

So

$$A = \begin{pmatrix} a & b \\ 0 & \frac{a}{\lambda} \end{pmatrix}, \quad B = \begin{pmatrix} x & -\frac{(\lambda x + x^{-1})b}{a(1-\lambda)} \\ 0 & x^{-1} \end{pmatrix},$$

where $a^2 = \lambda$. Similarly, since $b \neq 0$, we can conjugate by an element of $\operatorname{Stab}(\xi_\lambda)$ and assume $b = 1$. We get $S_{4,\lambda}^{1,b} \cong \mathbb{C}^* \times \{\pm\sqrt{\lambda}\}$ and $S_4^{1,b} \cong \mathbb{C}^* \times \mathbb{C} \setminus \{0, \pm i, \pm 1\}$. So

$$\begin{aligned} e(\overline{Y}_{4,\lambda}^{1,b}) &= 2(q-1)^2 \\ e(\overline{Y}_4^{1,b}) &= (q-5)(q-1)^2. \end{aligned}$$

Now, note that the $\mathbb{Z}_2$-action takes $S^{1,a}$ to $S^{1,b}$, interchanging the components. Therefore, to take account of the action we only need to consider one of them and

$$e\left(\left(S_4^{1,a} \cup S_4^{1,b}\right)/\mathbb{Z}_2\right) = (q-5)(q-1),$$

so

$$\begin{aligned} e(Y_4^{1,a} \cup Y_4^{1,b}) &= e\left(\left(S_4^{1,a} \cup S_4^{1,b}\right)/\mathbb{Z}_2\right) e(\mathrm{PGL}(2,\mathbb{C})) \\ &= (q^3 - q)(q-5)(q-1). \end{aligned}$$

- $Y_4^{1,c} := \{(A,B) \in Y_4 \mid b = c = 0\}$. The same equations, when $b = c = 0$, lead to

$$A = \begin{pmatrix} a & 0 \\ 0 & \frac{a}{\lambda} \end{pmatrix}, \quad B = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$$

where $\frac{a^2}{\lambda} = 1$. In this case, $\mathrm{Stab}\,(\xi_\lambda)$ stabilizes both A and B,

$$e(\overline{Y}_{4,\lambda}^{1,c}) = 2(q-1),$$

$$e(\overline{Y}_4^{1,c}) = (q-1)(q-5).$$

The $\mathbb{Z}_2$-action takes (a,x) to (a^{-1}, x^{-1}), where $a^2 = \lambda$, hence

$$\begin{aligned} e(S_4^{1,c}/\mathbb{Z}_2) &= e\left((\mathbb{C}^* \setminus \{\pm 1, \pm i\} \times \mathbb{C}^*)\,/\mathbb{Z}_2\right) \\ &= e(\mathbb{C}^* \setminus \{\pm 1, \pm i\})^+ e(\mathbb{C}^*)^+ + e(\mathbb{C}^* \setminus \{\pm 1, \pm i\})^- e(\mathbb{C}^*)^-. \end{aligned}$$

Now $\mu \mapsto \mu + \mu^{-1}$ maps $\mathbb{C}^*/\mathbb{Z}_2$ isomorphically to $\mathbb{C}$, from which it follows that $e(\mathbb{C}^*)^+ = q$ and $e\,(\mathbb{C}^* \setminus \{\pm 1, \pm i\})^+ = q - 3$. We get

$$\begin{aligned} e(S_4^{1,c}/\mathbb{Z}_2) &= (q-3)q + (-2)(-1) \\ &= q^2 - 3q + 2. \end{aligned}$$

Since $Y_4^{1,c} \cong \left(GL(2,\mathbb{C})/D \times S_4^{1,c}\right)/\mathbb{Z}_2$,

$$\begin{aligned} e(Y_4^{1,c}) &= e(GL(2,\mathbb{C})/D)^+ e(S_4^{1,c})^+ + e(GL(2,\mathbb{C})/D)^- e(S_4^{1,c})^- \\ &= q^2(q^2 - 3q + 2) + q(3 - 3q) \\ &= (q^3 - q)(q - 3). \end{aligned}$$

Adding up the E-polynomials of the three strata, we obtain

$$\begin{aligned} e(Y_4^1) &= e(Y_4^{1,a} \cup Y_4^{1,b} \cup Y_4^{1,c}) \\ &= (q^3 - q)(q-5)(q-1) + (q^3 - q)(q - 3) \\ &= (q^3 - q)(q^2 - 5q + 2). \end{aligned}$$

Cases $\{(A,B) \in Y_4 \mid bc \neq 0\}$**.**

We move on to the case where b and c are both different from zero. In this case, conjugating by an element of $\mathrm{Stab}\,(\xi_\lambda)$ we can assume that $b = 1$. The equations are

$$\begin{cases} z = -\lambda y c & (3.19) \\ (\lambda - 1)ay = \lambda x + t & (3.20) \\ (\lambda - 1)az + \lambda c(\lambda x + t) = 0 & (3.21) \\ c = \dfrac{a^2}{\lambda} - 1 & (3.22) \\ xt - yz = 1 & (3.23) \end{cases}$$

Substituting (3.22) into (3.19), and then into the other equations, we get

$$\begin{cases} z = y(\lambda - a^2) \\ t = (\lambda - 1)ay - \lambda x \\ c = \frac{a^2}{\lambda} - 1 \\ -\lambda x^2 - (1-\lambda)ayx - y^2(\lambda - a^2) = 1 \end{cases}$$

i.e. A and B have the form

$$A = \begin{pmatrix} a & 1 \\ \frac{a^2}{\lambda} - 1 & \frac{a}{\lambda} \end{pmatrix}, \quad B = \begin{pmatrix} x & y \\ y(\lambda - a^2) & (\lambda - 1)ay - \lambda x \end{pmatrix},$$

with the equation $\det B = 1$,

$$-\lambda x^2 - (1-\lambda)ayx - y^2(\lambda - a^2) = 1. \tag{3.24}$$

Also, since $c \neq 0$, we have $a^2 - \lambda \neq 0$. For every fixed $a \in \mathbb{C}^*$, $a^2 \neq \lambda$ we obtain the equation of a conic, so we have a conic bundle over the plane $\{(a, \lambda),\ a^2 \neq \lambda,\ \lambda \neq 0, \pm 1, a \in \mathbb{C}^*\}$.

We complete with the points at infinity to obtain a $\mathbb{P}^1$-bundle, and stratify according to the discriminant. First of all, we compute the $\mathbb{Z}_2$-action on this space, together with the slice fixing condition $b = 1$:

$$A = \begin{pmatrix} a & 1 \\ \frac{a^2}{\lambda} - 1 & \frac{a}{\lambda} \end{pmatrix} \overset{\mathbb{Z}_2}{\mapsto} \begin{pmatrix} \frac{a}{\lambda} & \frac{a^2}{\lambda} - 1 \\ 1 & a \end{pmatrix} \overset{\text{slice}}{\mapsto} \begin{pmatrix} \frac{a}{\lambda} & 1 \\ \frac{a^2}{\lambda} - 1 & a \end{pmatrix},$$

and also

$$B = \begin{pmatrix} x & y \\ y(\lambda - a^2) & (\lambda - 1)ay - \lambda x \end{pmatrix} \overset{\mathbb{Z}_2}{\mapsto} \begin{pmatrix} (\lambda - 1)ay - \lambda x & y(\lambda - a^2) \\ y & x \end{pmatrix}$$

$$\overset{\text{slice}}{\mapsto} \begin{pmatrix} -\lambda x + (\lambda - 1)ay & -\lambda y \\ y(\frac{a^2}{\lambda} - 1) & x \end{pmatrix},$$

so the action is

$$\begin{cases} a \mapsto \lambda^{-1} a \\ x \mapsto -\lambda x + (\lambda - 1)ay \\ y \mapsto \frac{y(\lambda - a^2)}{c} = -\lambda y. \end{cases}$$

Finally, the discriminant of (3.24) is

$$D := (\lambda - 1)^2 a^2 + 4\lambda(a^2 - \lambda) = ((\lambda + 1)a - 2\lambda)\,((\lambda + 1)a + 2\lambda)\,.$$

$Y_4^2 := \{(A, B) \in Y_4 \mid bc \neq 0, D = 0\}$.

We see that $D = 0$ if and only if:

$$a = \pm \frac{2\lambda}{\lambda + 1}.$$

Note that the condition $a^2 - \lambda \neq 0$ is automatically satisfied, since $\frac{4\lambda^2}{(\lambda+1)^2} - \lambda = -\frac{\lambda(\lambda-1)^2}{(\lambda+1)^2}$ and $\lambda \neq 0, 1$. Let us suppose that $a = \frac{2\lambda}{\lambda+1}$ (the other case is similar). The equation in this case is

$$-\lambda x^2 + (\lambda - 1)\frac{2\lambda}{\lambda+1}yx + y^2\left(\frac{4\lambda^2}{(\lambda+1)^2} - \lambda\right) = 1,$$

which, rearranging the terms, gives

$$\left(x - \frac{\lambda-1}{\lambda+1}y\right)^2 = -\frac{1}{\lambda},$$

two parallel lines. If we write $\mu := x - \frac{\lambda-1}{\lambda+1}y$, the equation is $\mu^2 = -\frac{1}{\lambda}$, $\mu \neq 0, \pm 1, \pm i$. We get an isomorphism between triples (λ, x, y) satisfying the equation and pairs (μ, y). Therefore, $S^2_{4,\lambda} \cong \frac{1}{\sqrt{-\lambda}} \times \mathbb{C}$ and $S^2_4 \cong \mathbb{C} \times \mathbb{C} \setminus \{0, \pm i, \pm 1\}$, so

$$e(S^2_{4,\lambda}) = 2q \quad e(S^2_4) = q(q-5).$$

The action of $\mathbb{Z}_2$ on μ is given by

$$\mu = x - \frac{\lambda-1}{\lambda+1}y \mapsto -\lambda x + (\lambda-1)ay + \frac{\lambda^{-1}-1}{\lambda^{-1}+1}\lambda y = \ldots = \frac{1}{\mu}.$$

If we write $Y = \frac{y}{\mu}$, the $\mathbb{Z}_2$-action leaves Y invariant. In addition, we get an isomorphism between pairs (y, μ), and pairs (Y, μ), $\mu \neq 0, \pm 1, \pm i$. Now, the $\mathbb{Z}_2$-action takes (Y, μ) to (Y, μ^{-1}), so the quotient is parametrized by pairs (Y, s), $s = \mu + \mu^{-1}$, $s \neq 0, \pm 2$.

We have arrived at an isomorphism $S^2_4/\mathbb{Z}_2 \cong \mathbb{C} \setminus \{0, \pm 2\} \times \mathbb{C}$. Doubling it to account for the other value of a, we get

$$e(Y^2_4) = (q^3 - q)2q(q-3).$$

Y^3_4, Y^4_4.

We deal now with the generic case: the bundle of non-degenerate conics. To be more precise, let us recall that the slice given by equations (3.19)-(3.23) gave us the single equation

$$-\lambda x^2 - (1-\lambda)ayx - y^2(\lambda - a^2) = 1,$$

where $a^2 - \lambda \neq 0$ and $D \neq 0$, i.e. $a \neq \pm\frac{2\lambda}{\lambda+1}$. This gives us a bundle of conics over the plane parametrized by (a, λ), minus the curves $a^2 - \lambda = 0$ and $a = \pm\frac{2\lambda}{\lambda+1}$. We can complete each conic with its points at infinity to obtain a $\mathbb{P}^1$-bundle, with equation

$$-\lambda x^2 - (1-\lambda)ayx - y^2(\lambda - a^2) = z^2.$$

This gives us the stratum Y^3_4, and the stratum given by the set of points at infinity, Y^4_4, which needs to be subtracted. The computation of these E-polynomials is very similar to that for the strata X_4^6, X_4^8 in [53], since the conic bundles are isomorphic. Their E-polynomials agree; they are

$$e(Y_4^3) = (q^3 - q)(q+1)(q^2 - 5q + 7)$$
$$e(Y_4^4) = (q^3 - q)(q^2 - 6q + 11).$$

Adding up all the polynomials of the stratification for Y_4, we obtain

$$\begin{aligned} e(Y_4) &= e(Y_4^1) + e(Y_4^2) + e(Y_4^3) - e(Y_4^4) \\ &= (q^3 - q)((q^2 - 5q + 2) + 2q(q-3) + (q+1)(q^2 - 5q + 7) - (q^2 - 6q + 11)) \\ &= (q^3 - q)(q^3 - 2q^2 - 3q - 2). \end{aligned}$$

Moreover, if we add up all the E-polynomials of the different strata, we get

$$\begin{aligned} e(Y) &= e(Y_0) + e(Y_1) + e(Y_2) + e(Y_3) + e(Y_4) \\ &= (q^3 - q)((q+4) + 1 + (q+1)(q-3) + (q^2 + 3q) + (q^3 - 2q^2 - 3q - 2)) \\ &= (q^3 - q)(q^3 - q) = (q^3 - q)^2, \end{aligned}$$

which equals $e(SL(2,\mathbb{C})^2)$, as expected.

3.3 E-polynomials of the character varieties of the Klein bottle

We proceed now to the computation of the E-polynomials of the character varieties associated to the fundamental group of the Klein bottle K. Using the stratification described in the previous section,

- $\mathcal{M}_{\mathrm{Id}}(K)$. We look at the different substrata in Y_0:
  - Y_0^1. The orbits in this stratum are S-equivalent to pairs

    $$\left(\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}, \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \right),$$

    where $\lambda \sim \lambda^{-1}$, since P_0 interchanges them. Writing $s = \lambda + \lambda^{-1}$, we see that the S-equivalence classes can be parametrized as points $(\pm 2, s)$. We therefore obtain two copies of $\mathbb{C}$.
  - Y_0^2. In this case, looking at the slice

    $$(A, B) = \left(\begin{pmatrix} \pm\lambda & 0 \\ 0 & \pm\lambda^{-1} \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right),$$

    where again $\lambda \sim \lambda^{-1}$, we obtain representatives for S-equivalence classes. They correspond to pairs $(s, 0)$, so we get another copy of $\mathbb{C}$, which intersects the previous two in the points $(2, 0), (-2, 0)$.

- Y_0^3, Y_0^4. The closures of the orbits given by the Jordan cases intersect Y_0^1. The corresponding points in the moduli space are just $(\pm 2, 0)$.

We conclude that the moduli space consists of three copies of $\mathbb{C}$, where one of them intersects the other two in the points $(2,0), (-2,0)$. Therefore

$$e(\mathcal{M}_{\mathrm{Id}}(K)) = 3q - 2.$$

- $\mathcal{M}_{-\mathrm{Id}}(K)$. In this case, the canonical forms for pairs $(A,B) \in Y_1$ are

$$(A,B) = \left(\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \right),$$

where $x \in \mathbb{C}^*$, which can be used as a parameter for the moduli space. Therefore

$$e(\mathcal{M}_{-\mathrm{Id}}(K)) = q - 1.$$

- $\mathcal{M}_{J_+}(K)$. In this case, we subdivided Y_2 in different strata according to the values of A. The case $a^2 \neq 1$ gives a contribution of $(q-3)(q-1)$ and when $a^2 = 1$, we obtain a contribution of $2((q-5)+2q) = 6q - 10$. Therefore

$$e(\mathcal{M}_{J_+}(K)) = (q-3)(q-1) + (6q-10) = q^2 + 2q - 7.$$

- $\mathcal{M}_{J_-}(K)$. In this case, we subdivided into two strata depending on whether $\operatorname{tr} A = 0$ or not. If $\operatorname{tr} A = 0$, we get a contribution of $2(q-1)$, and if $\operatorname{tr} A \neq 0$, we obtained a conic bundle over a punctured line, which gives us $q^2 + q + 2$. We obtain

$$e(\mathcal{M}_{J_-}(K)) = e(\overline{Y}_3)/e(U) = q^2 + 3q.$$

- $\mathcal{M}_{\xi_\lambda}(K)$, where $\xi_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, $\lambda \neq 0, \pm 1$. In this case, some orbits get identified when we consider the GIT-quotient. Looking at the different strata:

  - $Y_4^{1,a}, Y_4^{1,b}, Y_4^{1,c}$. We have slices

$$A = \begin{pmatrix} a & 0 \\ 1 & \frac{a}{\lambda} \end{pmatrix}, \; B = \begin{pmatrix} x & 0 \\ \frac{\lambda(\lambda x + x^{-1})}{(1-\lambda)a} & x^{-1} \end{pmatrix},$$

$$A = \begin{pmatrix} a & 1 \\ 0 & \frac{a}{\lambda} \end{pmatrix}, \; B = \begin{pmatrix} x & -\frac{\lambda(\lambda x + x^{-1})}{(1-\lambda)a} \\ 0 & x^{-1} \end{pmatrix},$$

$$A = \begin{pmatrix} a & 0 \\ 0 & \frac{a}{\lambda} \end{pmatrix}, \; B = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix},$$

  where in all cases $a^2 = \lambda$. We observe that, when considering S-equivalence classes, the points given by the first two strata are the same as the points given by the third. We obtain a contribution of $2(q-1)$ (given by the two copies parametrized by $x \in \mathbb{C}^*$).

– Y_4^2. For the slice $S_{4,\lambda}^2 \cong \pm\frac{1}{\sqrt{\lambda}} \times \mathbb{C}$, we get a contribution of $2q$, which we have to double to take into account the two components of $Y_{4,\lambda}^2$. The slice gives us a complete set of equivalence classes for the quotient.

– Y_4^3. $e(S_{4,\lambda}^3) = (q+1)(q-4)$ (see the stratum X_4^6 in [53]).

– Y_4^4. $e(S_{4,\lambda}^4) = (q-7)$, (see the stratum X_4^8, [53]).

Adding up, we get

$$e(\mathcal{M}_{\xi_\lambda}(K)) = 2(q-1) + 4q + (q+1)(q-4) - (q-7) = (q+1)^2.$$

This completes the proof of Theorem 3.1.1.

3.4 Relation with the orientable case

The Klein bottle K has the torus T as its orientable double cover, so there is a $2:1$ projection map $T \xrightarrow{\pi} K$. We have an exact sequence

$$0 \longrightarrow \pi_1(T) \xrightarrow{\pi_*} \pi_1(K) \longrightarrow \mathbb{Z}_2 \longrightarrow 0,$$

where the map π_* identifies $\pi_1(T)$ with the subgroup $H = \langle a, b^2 \rangle$ of $\pi_1(K) = \{a, b \mid abab^{-1} = e\}$. This natural map induces a map $\rho \mapsto \rho \circ \pi_*$ between representations of $\pi_1(K)$ and representations of $\pi_1(T)$, which descends to a map

$$\mathcal{M}_{\mathrm{Id}}(K) \xrightarrow{\pi^*} \mathcal{M}_{\mathrm{Id}}(T).$$

Note that the map π^* is $PGL(2,\mathbb{C})$-equivariant; this allows us to work with the several slices constructed in each case in the previous section. We analyze the map π^* with respect to the stratification given in Section 3.2 and we refer to [53] for the stratification and computation of $X_0 = \{(A,B) \in SL(2,\mathbb{C})^2 \mid [A,B] = \mathrm{Id}\}$. Basically, there are three strata in X_0: the case when either A or B is equal to $\pm\mathrm{Id}$, the case when they are both simultaneously diagonalizable, and the case when they can be simultaneously put into Jordan normal form. Hence:

Y_0^1. Recall that

$$Y_0^1 \cong \{\pm\mathrm{Id}\} \times SL(2,\mathbb{C}),$$

so given $(A,B) \in Y_0^1$, $(A,B^2) \in \{\pm\mathrm{Id}\} \times SL(2,\mathbb{C})$ again. Notice that if B is diagonalizable, B^2 is too, and that $B^2 = \mathrm{Id}$ for $B = \pm\mathrm{Id}$, $B^2 \sim J_+$ if B is of negative or positive Jordan type.

Y_0^2. The canonical forms for this stratum are

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

where $\lambda \sim \lambda^{-1}$ (P_0 acts conjugating the eigenvalues and leaving B invariant). Therefore, $B^2 = -\operatorname{Id}$ and A diagonalizable, which belongs to the stratum $SL(2,\mathbb{C}) \times \{-\operatorname{Id}\}$ of X_0.

Y_0^3. The slice consists of

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} \pm i & 0 \\ 0 & \mp i \end{pmatrix},$$

so that $B^2 = -\operatorname{Id}$ and A is of positive Jordan type. The map π^* takes this stratum to the stratum $SL(2,\mathbb{C}) \times \{-\operatorname{Id}\}$, where A is of positive Jordan type.

Y_0^4. In this case, the canonical forms are

$$A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} \pm i & 0 \\ 0 & \pm i \end{pmatrix},$$

so again $B^2 = -\operatorname{Id}$ and the image lies in $SL(2,\mathbb{C}) \times \{-\operatorname{Id}\}$.

We see that the map π^* is not injective or surjective at the level of representations, and that the image lies entirely in the set $(A,B) \in X_0$ such that A or B is equal to $\pm\operatorname{Id}$.

To study what happens at the moduli space level, recall that $\mathcal{M}_{\mathrm{Id}}(T)$ consists of S-equivalence classes which are of the form

$$\left(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} \right)$$

where $(\lambda,\mu) \in (\mathbb{C}^*)^2$ and $(\lambda,\mu) \sim (\lambda^{-1},\mu^{-1})$. Note that, unlike in the case of $\mathcal{M}_{\mathrm{Id}}(K)$, we cannot parametrize the set of S-equivalence clases by the traces, since the representation given by (λ,μ) need not be equivalent to (λ,μ^{-1}). So we will refer to pairs (λ,μ) under the mentioned equivalence relation when referring to representations in $\mathcal{M}_{\mathrm{Id}}(T)$.

With this description of $\mathcal{M}_{\mathrm{Id}}(T)$, we see that if $(\lambda,\mu) \neq (\pm 1,\mu), (\lambda,-1)$, the point is not in the image of π^*. Therefore π^* is not surjective.

Moreover, it is not injective. Looking at the S-equivalence classes given by the different strata:

- Y_0^1 gives us in the moduli space two copies of $\mathbb{C}$ given by pairs $(2,s)$ and $(-2,s)$ respectively. They are mapped to the classes $(\pm 1,\lambda^2)$, where λ is such that $s = \lambda + \lambda^{-1}$. In particular, the points given by $(2,0),(-2,0)$ are mapped to $(1,-1),(-1,-1)$ respectively. The map π^* is 2:1, branched over $(\pm 1,-1)$.

- Y_0^2 consists of the set of equivalence classes of the form $(s, 0)$ They are mapped to the S-equivalence classes in $\mathcal{M}_{\mathrm{Id}}(T)$ given by $(\lambda, -1)$. Note that, in this case, the map is $1:1$.

- Y_0^3 and Y_0^4 correspond to the two points $(\pm 2, 0)$, which, as already observed, map to $(\pm 1, -1)$.

Proposition 3.4.1. *Let φ be the involution on $\mathcal{M}_{\mathrm{Id}}(K)$ which takes a representation (A, B) to $(A, -B)$. Then the natural map π^* between $\mathcal{M}_{\mathrm{Id}}(K)$ and $\mathcal{M}_{\mathrm{Id}}(T)$ is constant on the $\mathbb{Z}_2$-orbits given by φ, and the induced map:*

$$\tilde{\pi}^* : \mathcal{M}_{\mathrm{Id}}(K)/\varphi \longrightarrow \mathcal{M}_{\mathrm{Id}}(T)$$

is injective.

Proof. It is obvious that π^* is constant on the $\mathbb{Z}_2$-orbits of ϕ. The injectivity of $\tilde{\pi}^*$ follows from the calculations preceding the statement of the proposition. $\square$

Remark 3.4.2. *Since $(A, B) \in Y_1 \Longrightarrow (A, B^2) \in X_0$, we have also a map $\pi^* : \mathcal{M}_{-\mathrm{Id}}(K) \longrightarrow \mathcal{M}_{\mathrm{Id}}(T)$. Using the isomorphism $\mathcal{M}_{-\mathrm{Id}}(K) \cong \mathbb{C}^*$, we see that π^* maps $x \in \mathbb{C}^*$ to $(i, x^2) \in \mathcal{M}_{\mathrm{Id}}(T)$. One cannot extend π^* to the other moduli spaces in this way, since a stratum Y_i may not map to a single stratum X_j.*

The torus involution

There is an involution $\tau : T \longrightarrow T$ of the torus which corresponds to the generator of the group of deck transformations $Deck(T) \cong \mathbb{Z}_2$ of the covering $\pi : T \longrightarrow K$. Identifying $\pi_1(T, p_0)$ and $\pi_1(T, \tau(p_0))$ (we will not make further references to the base point), this involution induces an automorphism of the fundamental group $\pi_1(T)$:

$$\begin{aligned} \tau_* : \pi_1(T) &\longrightarrow \pi_1(T) \\ a &\longrightarrow a^{-1} \\ b &\longrightarrow b \end{aligned}$$

(observe that $[a^{-1}, b] = a^{-1}(bab^{-1}) = a^{-1}a = e$) which again induces another map between representations by composition on the left. It is compatible with the action of G by conjugation, so we obtain

$$\tau^* : \mathcal{M}_{\mathrm{Id}}(T) \longrightarrow \mathcal{M}_{\mathrm{Id}}(T)$$

Note that the point of $\mathcal{M}_{\mathrm{Id}}(T)$ corresponding to the representation ρ is invariant under τ^* if and only if there exists $g \in SL(2, \mathbb{C})$ such that $g\tau^*(\rho)g^{-1} = \rho$.

Proposition 3.4.3. *The image of the map $\pi^* : \mathcal{M}_{\mathrm{Id}}(K) \longrightarrow \mathcal{M}_{\mathrm{Id}}(T)$ lies in the τ-invariant part $(\mathcal{M}_{\mathrm{Id}}(T))^\tau$. The induced map*

$$\pi'^* : \mathcal{M}_{\mathrm{Id}}(K) \longrightarrow \mathcal{M}_{\mathrm{Id}}(T)^\tau$$

is not surjective.

Proof. The first statement can be checked stratum by stratum. The image of π'^* does not contain the points of $\mathcal{M}_{\mathrm{Id}}(T)^\tau$ given by $(\lambda, 1)$ for $\lambda \neq \pm 1$. $\square$

Remark 3.4.4. *In [49], which is set in a much more general context, a representation is called good if it is reductive and is stabilised only by the centre $Z(G)$. According to [49, Theorem 1.2], adapted to our situation, there is a Galois double covering map from $\mathcal{M}_{\mathrm{Id}}(K)^{good}_\pi$ to $\mathcal{M}_{\mathrm{Id}}(T)^{good}$. (Here $\mathcal{M}_{\mathrm{Id}}(K)^{good}_\pi$ corresponds to representations whose pullback to $\pi_1(T)$ is good.) In fact, these spaces are both empty, although there do exist good representations of $\pi_1(K)$, namely those corresponding to the stratum Y_0^2. As we have seen, the map π^* is actually 1:1 on this stratum, although it is 2:1 on the other strata, where the representations are not good.*

Remark 3.4.5. *We already saw that $\mathcal{M}_{-\mathrm{Id}}(K)$ was mapped under π^* to $\mathcal{M}_{\mathrm{Id}}(T)$, more precisely, to the set of points (i, x^2). Notice that in this case it is not mapped to the invariant part $\mathcal{M}_{\mathrm{Id}}(T)^\tau$.*

3.5 $SL(2,\mathbb{C})$-character variety of the connected sum of three projective planes

In this and subsequent sections in this chapter, we consider the case of the connected sum of three real projective planes, $\#^3 P^2$. We write $\Sigma = \#^3 P^2$ and let $\tilde{\Sigma}$ denote its orientable double cover, homeomorphic to the genus 2 surface. We are interested in the $SL(2,\mathbb{C})$-character variety

$$\mathcal{M}_{\mathrm{Id}}(\Sigma) = \{(A,B,C) \in SL(2,\mathbb{C})^3 \mid [A,B] = C^2\}//G,$$

and also in the additional set of character varieties $\mathcal{M}_{-\mathrm{Id}}(\Sigma), \mathcal{M}_{J_+}(\Sigma), \mathcal{M}_{J_-}(\Sigma)$ and $\mathcal{M}_{\xi_\lambda}(\Sigma)$. We consider the $\mathrm{PGL}(2,\mathbb{C})$-equivariant map

$$\begin{aligned} \tilde{g} : SL(2,\mathbb{C})^3 &\longrightarrow SL(2,\mathbb{C}) \\ (A,B,C) &\longrightarrow [A,B]C^{-2} \end{aligned}$$

and focus in this section on $M_0 := \tilde{g}^{-1}(\mathrm{Id}) = \{(A,B,C) \mid [A,B] = C^2\}$, the space of representations of $\pi_1(\Sigma)$ into $SL(2,\mathbb{C})$. Notice that in this case we will have to take a

GIT quotient since there are reducible and non-reducible orbits under the $\mathrm{PGL}(2,\mathbb{C})$-action whose closures intersect. To compute its E-polynomial, we consider the stratification

- $M_0^0 = \{(A,B,C) \in SL(2,\mathbb{C})^3 \mid [A,B] = C^2 = \mathrm{Id}\}$.
- $M_0^1 = \{(A,B,C) \in SL(2,\mathbb{C})^3 \mid [A,B] = C^2 = -\mathrm{Id}\}$.
- $M_0^2 = \{(A,B,C) \in SL(2,\mathbb{C})^3 \mid [A,B] = C^2 \sim J_+\}$.
- $M_0^3 = \{(A,B,C) \in SL(2,\mathbb{C})^3 \mid [A,B] = C^2 \sim J_-\}$.
- $M_0^4 = \left\{(A,B,C) \in SL(2,\mathbb{C})^3 \mid [A,B] = C^2 \sim \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \text{for some } \lambda \neq 0, \pm 1\right\}$,

so that

$$M_0 = \sqcup_{i=0}^4 M_0^i.$$

We will also write $\overline{M}_0^i$ to refer to the fixed conjugacy Hom spaces:

- $\overline{M}_0^2 = \{(A,B,C) \in SL(2,\mathbb{C}) \mid [A,B] = C^2 = J_+\}$,
- $\overline{M}_0^3 = \{(A,B,C) \in SL(2,\mathbb{C}) \mid [A,B] = C^2 = J_-\}$,
- $\overline{M}_0^4 = \left\{(A,B,C,\lambda) \in SL(2,\mathbb{C})^3 \times \mathbb{C}^* \setminus \{\pm 1\} \mid [A,B] = C^2 = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}\right\}$.

Proceeding case by case:

- $M_0^0 = \{(A,B,C) \in SL(2,\mathbb{C})^3 \mid [A,B] = C^2 = \mathrm{Id}\} \cong X_0 \times \{\pm \mathrm{Id}\}$, so

$$e(M_0^0) = 2e(X_0) = 2q^4 + 8q^3 - 2q^2 - 8q.$$

- $M_0^1 = \{(A,B,C) \in SL(2,\mathbb{C})^3 \mid [A,B] = C^2 = -\mathrm{Id}\}$. Note that the equation $C^2 = -\mathrm{Id}$ has a continuous family of solutions in $SL(2,\mathbb{C})$, given by the two strata C_i, according to whether $c = 0$ or $c \neq 0$:
  - $C_1 = \left\{\begin{pmatrix} \pm i & b \\ 0 & \mp i \end{pmatrix}\right\} \cong \{\pm i\} \times \mathbb{C}$.
  - $C_2 = \left\{\begin{pmatrix} a & \frac{-1-a^2}{c} \\ c & -a \end{pmatrix}\right\} \cong \mathbb{C} \times \mathbb{C}^*$.

  C_1 and C_2 are disjoint, so

$$M_0^1 \cong X_1 \times (C_1 \sqcup C_2) = X_1 \times C_1 \sqcup X_1 \times C_2,$$

  and therefore

$$e(M_0^1) = e(X_1)(e(C_1) + e(C_2)) = q^5 + q^4 - q^3 - q^2.$$

- $M_0^2 = \{(A,B,C) \in SL(2,\mathbb{C})^3 \mid [A,B] = C^2 \sim J_+\}$. The equation $C^2 = J_+$ has two solutions in $SL(2,\mathbb{C})$, namely
$$C^2 = J_+ \Rightarrow C = \pm \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix} := \pm\sqrt{J_+},$$
so $\overline{M}_0^2 \cong \overline{X}_2 \times \{\pm\sqrt{J_+}\}$ and $e(\overline{M}_0^2) = 2e(\overline{X}_2) = 2(q^3 - 2q^2 - 3q) = 2q^3 - 4q^2 - 6q$.
To compute M_0^2, we use the fibration
$$U \longrightarrow GL(2,\mathbb{C}) \times \overline{M}_0^2 \longrightarrow M_0^2$$
from which we deduce
$$e(M_0^2) = e(GL(2,\mathbb{C})/U)e(\overline{M}_0^2) = 2q^5 - 4q^4 - 8q^3 + 4q^2 + 6q.$$
- $M_0^3 = \{(A,B,C) \in SL(2,\mathbb{C})^3 \mid [A,B] = C^2 \sim J_-\}$. If we try to compute $\overline{M}_0^3$, we see that the equation $C^2 = J_-$ has no solutions in $SL(2,\mathbb{C})$. Therefore, $\overline{M}_0^3 = \emptyset$, which also implies that $M_0^3 = \emptyset$.
- $M_0^4 = \left\{(A,B,C) \in SL(2,\mathbb{C})^3 \mid [A,B] = C^2 \sim \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \text{for some } \lambda \neq 0, \pm 1\right\}$.

To compute its E-polynomial, let us consider the fibration:
$$\overline{M}_{0,\lambda}^4 \longrightarrow \overline{M}_0^4 \longrightarrow \mathbb{C}^* \setminus \{\pm 1\} \tag{3.25}$$
where $\overline{M}_{0,\lambda}^4 = \{(A,B,C) \in SL(2,\mathbb{C})^3 \mid [A,B] = C^2 = \xi_\lambda\}$. It is a locally trivial fibration with fibre isomorphic to $\overline{X}_{4,\lambda} \times \pm\sqrt{\lambda}$, since the only solutions in $SL(2,\mathbb{C})$ to $C^2 = \xi_\lambda$ are the two diagonal matrices whose eigenvalues are the square roots of λ. We write $\pm\sqrt{\lambda} \longrightarrow \overline{Z} \longrightarrow \mathbb{C}^* \setminus \{\pm 1\}$ for this square root fibration, where $\overline{Z} = \{C \in SL(2,\mathbb{C}) \mid C^2 = \xi_\lambda\}$.

There is a $\mathbb{Z}_2$-action on the three spaces $\overline{M}_0^4$, $\overline{X}_4$ and $\overline{Z}$ that covers the action $\lambda \mapsto \lambda^{-1}$ on the base, giving rise to fibrations $\overline{M}_0^4/\mathbb{Z}_2$, $\overline{X}_4/\mathbb{Z}_2$ and $\overline{Z}/\mathbb{Z}_2$ over $(\mathbb{C}^* \setminus \{\pm 1\})/\mathbb{Z}_2 \cong \mathbb{C} \setminus \{\pm 2\}$ respectively. Let us also write $H_1(\mathbb{C}^* \setminus \{\pm 1\}) = \langle \gamma_0, \gamma_{-1}, \gamma_1 \rangle$ and $H_1(\mathbb{C} \setminus \{\pm 2\}) = \langle \nu_2, \nu_{-2} \rangle$. Note that the quotient map $\lambda \mapsto \lambda + \lambda^{-1}$ maps $\gamma_{\pm 1}$ to $2\nu_{\pm 2}$ and γ_0 to $\nu_2 + \nu_{-2}$.

The Hodge monodromy representation of $R(\overline{M}_0^4/\mathbb{Z}_2)$ can be used to recover the E-polynomials of $\overline{M}_0^4$, $\overline{M}_0^4/\mathbb{Z}_2$ and M_0^4, using Corollary 2.3.6. From the previous description of $\overline{M}_0^4$, we get
$$R(\overline{M}_0^4/\mathbb{Z}_2) = R(\overline{X}_4/\mathbb{Z}_2) \otimes R(\overline{Z}/\mathbb{Z}_2).$$
The Hodge monodromy representation $R(\overline{X}_4/\mathbb{Z}_2)$ was computed in [53],
$$R(\overline{X}_4/\mathbb{Z}_2) = q^3 T - 3qS_2 + 3q^2 S_{-2} - S_0 \in R(\mathbb{Z}_2 \times \mathbb{Z}_2)[q], \tag{3.26}$$

where $S_2, S_{-2}, S_0 := S_2 \otimes S_{-2}$ denote the representations which are trivial on ν_2, ν_{-2} and $\gamma_0 = \nu_2 + \nu_{-2}$ respectively.

To compute $R(\overline{Z}/\mathbb{Z}_2)$, let us start by looking at $R(\overline{Z})$. The fibre $\overline{Z}_\lambda$ consists of two points on which the monodromy acts as follows: the actions of γ_1, γ_{-1} are trivial and the action of γ_0 interchanges them. It is esentially the monodromy described by the square root map in $\mathbb{C}$,

$$R(\overline{Z}) = T + N \in R(\mathbb{Z}_2)[q],$$

where T, N are the trivial and non trivial representations of $\mathbb{Z}_2$. To study $R(\overline{Z}/\mathbb{Z}_2)$, note that $R(\overline{Z}/\mathbb{Z}_2)$ reduces to $R(\overline{Z})$ under the mentioned double cover $\mathbb{C}^* \setminus \{\pm 1\} \to \mathbb{C} \setminus \{\pm 2\}$, so the monodromy is of order two around ν_2, ν_{-2}, and hence $R(\overline{Z}/\mathbb{Z}_2) \in R(\mathbb{Z}_2 \times \mathbb{Z}_2)[q]$. With that in mind, we may proceed as in [53] and fix a determination of the logarithm with branch $\{iy : y < 0\}$ and lift the paths ν_2, ν_{-2} to paths in $\mathbb{C}^* \setminus \{\pm 1\}$. We see that ν_2 acts trivially on the fibre, ν_{-2} permutes the points and therefore γ_0 does the same (the action of γ_0 is given by the composition of both actions). So if we write $R(\overline{Z}/\mathbb{Z}_2) = aT + bS_2 + cS_{-2} + dS_0$,

$$\begin{aligned} &e(F) = 2, \quad e(F)^{\mathrm{inv}} = a = 1, \quad e(F)^{\nu_2} = a + b = 2, \\ &e(F)^{\nu_{-2}} = a + c = 1, \quad e(F)^{\gamma_0} = a + d = 1, \end{aligned}$$

from where we deduce that $a = b = 1$, $c = d = 0$, and therefore

$$R(\overline{Z}/\mathbb{Z}_2) = T + S_2.$$

We have obtained that

$$\begin{aligned} R(\overline{M}_0^4/\mathbb{Z}_2) &= R(\overline{Z}/\mathbb{Z}_2) \otimes R(\overline{X}_4/\mathbb{Z}_2) \\ &= (T + S_2) \otimes (q^3 T - 3qS_2 + 3q^2 S_{-2} - S_0) \\ &= (q^3 - 3q)T + (q^3 - 3q)S_2 + (3q^2 - 1)S_{-2} + (3q^2 - 1)S_0. \end{aligned}$$

If we write $R(\overline{M}_0^4/\mathbb{Z}_2) = a''T + b''S_2 + c''S_{-2} + d''S_0$ and apply Proposition 2.3.6, we have

$$\begin{aligned} e(\overline{M}_0^4) &= (q-3)(a'' + d'') - 2(b'' + c'') = q^4 - 2q^3 - 18q^2 + 14q + 5 \\ e(\overline{M}_0^4/\mathbb{Z}_2) &= (q-2)a'' - (b'' + c'' + d'') = q^4 - 9q^2 - 3q^3 + 9q + 2 \end{aligned}$$

and also applying Proposition 2.4.3,

$$\begin{aligned} e(M_0^4) &= q(q-1)e(\overline{M}_0^4/\mathbb{Z}_2) + qe(\overline{M}_0^4) \\ &= q(q^2 - 2q - 1)a'' - q(q+1)(b'' + c'') - 2qd'' \\ &= q^6 - 3q^5 - 8q^4 + 7q^2 + 3q. \end{aligned}$$

Computation of the E-polynomial of the character variety

To compute the E-polynomial of the moduli space $\mathcal{M}_{Id}(\Sigma)$, we need to take a GIT quotient since there are reducible and non reducible orbits. Looking at the description given in the previous section for the set of representations:

- *Reducible orbits.* All orbits in M_0 are reducible since $[A, B] = \mathrm{Id}$ implies that A, B are upper triangular and $C = \pm\,\mathrm{Id}$. All orbits in M_0^i are reducible too, since looking at the matrices in $\overline{M}_0^i$ they are all upper triangular. Every reducible orbit is S-equivalent to a triple of the form
$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}, \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix},$$
where $(\lambda, \mu, \pm 1) \sim (\lambda^{-1}, \mu^{-1}, \pm 1)$. So $\mathcal{R} \cong \sqcup_{i=1}^{2} (\mathbb{C}^* \times \mathbb{C}^*)/\mathbb{Z}_2$ and therefore
$$e(\mathcal{R}) = 2(q^2 + 1).$$

- *Irreducible orbits.* The orbits in M_1, M_4 are all irreducible. Their E-polynomial is
$$e(\mathcal{I}) = (e(M_1) + e(M_4))/e(\mathrm{PGL}(2, \mathbb{C})) = q^3 - 2q^2 - 6q - 3.$$

Adding the contributions, we obtain
$$e(\mathcal{M}_{\mathrm{Id}}(\Sigma)) = e(\mathcal{R}) + e(\mathcal{I}) = q^3 - 6q - 1.$$

Relation with the orientable case

Recall that there is a short exact sequence
$$0 \longrightarrow \pi_1(\tilde{\Sigma}) \xrightarrow{\pi_*} \pi_1(\Sigma) \longrightarrow \mathbb{Z}_2 \longrightarrow 0,$$
where π_* is the map between fundamental groups induced by the projection π. If we choose presentations
$$\pi_1(\tilde{\Sigma})) = \langle x, y, \hat{x}, \hat{y} \mid [x, y][\hat{x}, \hat{y}] = e \rangle,$$
and
$$\pi_1(\Sigma) = \langle a, b, c \mid [a, b] = c^2 \rangle,$$
then π_* can be described as
$$\begin{aligned} \pi_* : \pi_1(\tilde{\Sigma}) &\mapsto \pi_1(\Sigma) \\ x &\mapsto a \\ y &\mapsto b \\ \hat{x} &\mapsto cbc^{-1} \\ \hat{y} &\mapsto cac^{-1} \end{aligned}$$

The map π_* induces a contravariant map π^* between representations that descends to a map between moduli spaces

$$\mathcal{M}_{\mathrm{Id}}(\Sigma) \xrightarrow{\pi^*} \mathcal{M}_{\mathrm{Id}}(\tilde{\Sigma}).$$

We are also interested in the map induced on $\pi_1(\tilde{\Sigma})$ by the involution τ, the generator of the group of deck transformations of the orientable double cover $\tilde{\Sigma}$. After identifying $\pi_1(\tilde{\Sigma}, p) \cong \pi_1(\tilde{\Sigma}, \tau(p))$, this map can be described as

$$\begin{aligned} \tau_* : \pi_1(\tilde{\Sigma}) &\mapsto \pi_1(\tilde{\Sigma}) \\ (x, y, \hat{x}, \hat{y}) &\mapsto (\hat{y}, \hat{x}, [x, y]y[y, x], [x, y]x[y, x]) \end{aligned}$$

We characterize in this section the set of points that are in the image of π^*, which lie inside the τ-invariant part $(\mathcal{M}_{\mathrm{Id}}(\tilde{\Sigma}))^\tau$ of the total moduli space $\mathcal{M}_{\mathrm{Id}}(\tilde{\Sigma})$ as expected. In order to do so, let us first characterize the set of τ-invariant representations. By definition, given $\rho \in \mathrm{Hom}\,(\pi_1(\tilde{\Sigma}), SL(2,\mathbb{C}))$, ρ is τ-invariant if and only if there exists $g \in SL(2,\mathbb{C})$ such that

$$\tau^*(\rho) = g\rho g^{-1}.$$

If we write $\rho = (X, Y, \hat{X}, \hat{Y})$ and $\gamma = [X, Y] = [\hat{Y}, \hat{X}]$, this will happen if and only if there exists $g \in SL(2,\mathbb{C})$ satisfying

$$(\hat{Y}, \hat{X}, \gamma Y \gamma^{-1}, \gamma X \gamma^{-1}) = (gXg^{-1}, gYg^{-1}, g\hat{X}g^{-1}, g\hat{Y}g^{-1}). \tag{3.27}$$

The following conditions are therefore necessary and sufficient:

$$\hat{X} = gYg^{-1} \tag{3.28}$$

$$\hat{Y} = gXg^{-1} \tag{3.29}$$

$$\gamma^{-1}g^2 \in \mathrm{Stab}\,(X, Y). \tag{3.30}$$

Notice that conditions (3.28) and (3.29) imply that $g \in \mathrm{Stab}\ \gamma$ since $\gamma = [\hat{Y}, \hat{X}] = [gXg^{-1}, gYg^{-1}] = g\gamma g^{-1}$.

To check which orbits in $Y' := \mathrm{Hom}\,(\pi_1(\tilde{\Sigma}), SL(2,\mathbb{C}))$ are τ-invariant, we will use again the stratification of $\mathrm{Hom}\,(\pi_1(\tilde{\Sigma}), SL(2,\mathbb{C}))$ described in [53], where

$$Y' = \bigsqcup Y'_i$$

and

$$Y'_i = \{(X, Y, \hat{X}, \hat{Y}) \mid [X, Y] = [\hat{Y}, \hat{X}] \sim \alpha_i\},$$

where $\alpha_i = \mathrm{Id}, -\mathrm{Id}, J_+, J_-, \xi_\lambda$ for $i = 0, \ldots, 4$ respectively. Notice also that the map π^* maps the stratum M_0^i of M_0 to the stratum Y'_i of Y' for all $i = 0 \ldots 4$. Since we will be dealing with $\mathrm{Stab}\,(X, Y)$ because of condition (3.30), we recall the following definition.

Definition 3.5.1. *A representation* $\rho \in \operatorname{Hom}(\pi_1(\tilde{\Sigma}), G)$ *is simple if* $\operatorname{Stab}(\rho) = Z(G)$.

In our particular case where $G = SL(2,\mathbb{C})$, every irreducible representation is simple. Using conditions (3.28), (3.29) and (3.30), we describe the τ-invariant part of every stratum:

- $Y_0' := \{(X, Y, \hat{X}, \hat{Y}) \in SL(2,\mathbb{C})^4 \mid [X, Y] = [\hat{Y}, \hat{X}] = Id\}$. Since $\gamma = \mathrm{Id}$, condition (3.30) is rewritten as $g^2 \in \operatorname{Stab}(X, Y)$. Notice that since $[X, Y] = \mathrm{Id}$, both matrices have a common eigenvector. So $\operatorname{Stab}(X, Y) = D$ if they are both diagonalizable or $\operatorname{Stab}(X, Y) = U$ if they are of Jordan type. In either case, $g^2 \in \operatorname{Stab}(X, Y)$ implies that $g \in \operatorname{Stab}(A, B)$, since the square root of a diagonal matrix is diagonal and the same holds for the subgroup U. The set of τ-invariant representations is of the form

$$Y_0'^{\tau} = \{(X, Y, Y, X) \in SL(2,\mathbb{C})^4 \mid [X, Y] = \mathrm{Id}\}.$$

- $Y_2' := \{(X, Y, \hat{X}, \hat{Y}) \in SL(2,\mathbb{C})^4 \mid [X, Y] = [\hat{Y}, \hat{X}] \sim J_+\}$. Looking at the slice for X_2 in [53],

$$X = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, \quad Y = \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix},$$

where $yx(a^2 - 1) - ba(x^2 - 1) = 1$, $(a, x) \neq (\pm 1, \pm 1)$. Because of this fact, we see that all representations in $\overline{X}_2$ are reducible and simple. Therefore $\operatorname{Stab}(X, Y) = \pm \mathrm{Id}$ and condition (3.30) implies that

$$g^2 = \pm J_+ = \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The equation $g^2 = -J_+$ has no solutions, so we are left with the positive Jordan case, where $g = \pm \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}$. We write g_1 and g_2 for these solutions. We obtain that τ-invariant representations are of the form

$$Y_2'^{\tau} := \{(X, Y, g_i Y g_i^{-1}, g_i X g_i^{-1}), i = 1, 2 \mid (X, Y) \in \overline{X}_2\}.$$

The points in Y_0' and Y_2', which form the reducible locus, get identified when considering the GIT quotient since the closures of their orbits intersect. Looking at the τ-invariant part in these two spaces we see that at the moduli space level

$$\mathcal{M}_{\mathrm{Id}}(\tilde{\Sigma})^{red,\tau} = \{(\lambda, \mu, \mu, \lambda) \mid (\lambda, \mu, \mu, \lambda) \sim (\lambda^{-1}, \mu^{-1}, \mu^{-1}, \lambda^{-1}), \lambda, \mu \in \mathbb{C}^*\}. \quad (3.31)$$

- Y_1', Y_3', Y_4'. All these strata are irreducible and therefore simple, so (3.30) becomes

$$g^2 = \pm\, \gamma, \quad (3.32)$$

where $\gamma = -\,\mathrm{Id}, J_-$ or ξ_λ respectively. For such $g \in G$, (3.28), (3.29) and (3.32) imply that the τ-invariant part consists of the following orbits, which we subdivide in $+$ and $-$ according to the sign in equation (3.32):

$$\begin{aligned}
(Y_1')^\tau &= \{(X, Y, g_i Y g_i^{-1}, g_i X g_i^{-1}) \mid g_i^2 = \pm\,\mathrm{Id}, [X, Y] = -\,\mathrm{Id}\} \\
&= (Y_1')^{\tau,+} \cup (Y_1')^{\tau,-}, \qquad (3.33) \\
(Y_3')^\tau &= \{(X, Y, g_i B g_i^{-1}, g_i X g_i^{-1}) \mid g_i^2 = \pm J_-, [X, Y] = J_-\} \\
&= (Y_3')^{\tau,+} \cup (Y_3')^{\tau,-}, \qquad (3.34) \\
(Y_4')^\tau &= \{(X, Y, g_i Y g_i^{-1}, g_i X g_i^{-1}) \mid g_i^2 = \pm\xi_\lambda, [X, Y] = \xi_\lambda\} \\
&= (Y_4')^{\tau,+} \cup (Y_4')^{\tau,-}. \qquad (3.35)
\end{aligned}$$

Observe that $(Y_3')^{\tau,+}$ is empty, since there are no solutions to the equation $g^2 = J_-$. In all the other cases, for each choice of sign there are two $g_i \in SL(2, \mathbb{C}), i = 1, 2$ such that $g_i^2 = \gamma = \pm[X, Y]$. To obtain representatives for the quotients under the $\mathrm{PGL}(2, \mathbb{C})$-action, all is needed is to take (X, Y) belonging to the respective slices constructed for the $SL(2, \mathbb{C})$-action.

We turn now to look at π^*.

Proposition 3.5.2. *The map π^* between moduli spaces*

$$\mathcal{M}_{\mathrm{Id}}(\Sigma) \xrightarrow{\pi^*} (\mathcal{M}_{\mathrm{Id}}(\tilde{\Sigma}))^\tau,$$

is a 2:1 covering onto its image, $(\mathcal{M}_{\mathrm{Id}}(\tilde{\Sigma}))^{\tau,red} \sqcup (\mathcal{M}_{\mathrm{Id}}(\tilde{\Sigma}))^{\tau,+}$.

Proof. The map π^* takes a representation $(A, B, C) \in \mathrm{Hom}\,(\pi_1(\Sigma), SL(2, \mathbb{C}))$ to the representation $(A, B, CBC^{-1}, CAC^{-1}) \in \mathrm{Hom}\,(\pi_1(\tilde{\Sigma}), SL(2, \mathbb{C}))$. Since $[A, B] = C^2$, if we make $g = C$, we see that $\pi^*(\rho)$ satisfies conditions (1)-(3). So the image of π^* lies in the τ-invariant part.

The reducible orbits in $\mathcal{M}_{\mathrm{Id}}(\Sigma)$ arise from representations in M_0^0,M_0^2. Looking at the stratifications, notice that the set of reducible representations in $\mathcal{M}_{\mathrm{Id}}(\Sigma)$ is mapped to the set of reducible representations in $\mathcal{M}_{\mathrm{Id}}(\Sigma)$. The same is true for the irreducible ones.

A reducible $(\lambda, \mu, \pm 1) \in \mathcal{M}_{\mathrm{Id}}(\Sigma)$ is mapped by the map π^* to the representation

$$(\lambda, \mu, \mu, \lambda) \in \mathcal{M}_{\mathrm{Id}}(\tilde{\Sigma}),$$

which by (3.31) corresponds to all the τ-invariant reducible representations in $\mathcal{M}_{\mathrm{Id}}(\tilde{\Sigma})$ (arising from Y_0', Y_2'). The map is therefore surjective and 2:1 on the reducible part.

For the irreducible locus, the τ-invariant part was described in (3.33), (3.34) and (3.35) as representations

$$\{(X, Y, g_i Y g_i^{-1}, g_i X g_i^{-1}) \mid g_i^2 = \pm[X, Y]\}.$$

Now given $(A, B, C) \in \mathcal{M}_{\mathrm{Id}}(\Sigma)$ (recall that $[A, B] = C^2$) we see that the image is

$$(A, B, CBC^{-1}, CAC^{-1}) \in X_i^{\tau,+} \subset (\mathcal{M}_{\mathrm{Id}}(\tilde{\Sigma}))^{\tau},$$

so the map is not surjective. If we restrict to the positive part, then

$$\mathcal{M}_{\mathrm{Id}}^{irr}(\Sigma) \xrightarrow{\pi^*} (\mathcal{M}_{\mathrm{Id}}^{irr})^{\tau,+}$$

is surjective and 2:1, since the representation given by $(X, Y, gYg^{-1}, gXg^{-1})$, where $g^2 = [X, Y]$ has two preimages: (X, Y, g), $(X, Y, -g) \in \mathcal{M}_{\mathrm{Id}}(\Sigma)$ (these two representations are not equivalent even if $g \overset{P}{\sim} -g$, since the pair (X, Y) is simple). This completes the proof. $\square$

Remark 3.5.3. *Proposition 3.5.2 agrees with Theorem 1.2 in [49], where it is asserted that there is a Galois double covering map from* $\mathcal{M}_{\mathrm{Id}}(\Sigma)_{\pi}^{good}$ *to* $\mathcal{N}_0^{good} \subset \mathcal{M}(\tilde{\Sigma})^{good,\tau}$.

In our notation, $\mathcal{N}_0^{good}$ *corresponds to* $\mathcal{M}_{\mathrm{Id}}(\tilde{\Sigma})^{\tau,+}$ *(the irreducible part is simple, which is called good in [49]). We also get a 2:1 covering on the reducible locus* $\mathcal{M}(\Sigma)^{red} \mapsto \mathcal{M}(\tilde{\Sigma})^{\tau,red}$. *As we saw in section 3.4, these statements fail for the Klein bottle and the genus* 1 *case.*

3.6 E-polynomial of the twisted $SL(2,\mathbb{C})$-character variety of Σ

We focus now on the twisted $SL(2,\mathbb{C})$-character variety of Σ,

$$\mathcal{M}_{-\mathrm{Id}}(\Sigma) = M_1 / \mathrm{PGL}(2,\mathbb{C}),$$

where $M_1 = \tilde{g}^{-1}(-\mathrm{Id})$. We stratify in different cases according to the value of $[A, B]$:

$$M_1^0 = \{(A, B, C) \in SL(2,\mathbb{C})^3 \mid [A, B] = -C^2 = \mathrm{Id}\},$$

$$M_1^1 = \{(A, B, C) \in SL(2,\mathbb{C})^3 \mid [A, B] = -C^2 = -\mathrm{Id}\},$$

$$M_1^2 = \{(A, B, C) \in SL(2,\mathbb{C})^3 \mid [A, B] = -C^2 \sim J_+\},$$

$$M_1^3 = \{(A, B, C) \in SL(2,\mathbb{C})^3 \mid [A, B] = -C^2 \sim J_-\},$$

$$M_1^4 = \{(A, B, C) \in SL(2,\mathbb{C})^3 \mid [A, B] = -C^2 \sim \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \text{for some } \lambda \neq 0, \pm 1\}.$$

Since $M_1^0 \cong X_0 \times (C_1 \sqcup C_2)$ and $M_1^1 \cong X_1 \times \{\pm \mathrm{Id}\}$, we obtain

$$e(M_1^0) = e(X_0)(e(C_1) + e(C_2)) = (q^4 + 4q^3 - q^2 - 4q)(q^2 + q),$$

$$e(M_1^1) = 2e(X_1) = 2(q^3 - q).$$

M_1^2 is empty, since there are no solutions to the equation $C^2 = -J_+$. The E-polynomial of M_1^3 is computed as the E-polynomial of M_0^3 in Section 3.5,

$$e(M_1^3) = e(GL/U)e(\overline{M}_1^3) = (q^2 - 1)2(q^3 + 3q^2) = 2q^5 + 6q^4 - 2q^3 - 6q^2.$$

Finally, to compute M_1^4, note that we have a fibration

$$\overline{M}_{1,\lambda}^4 \longrightarrow \overline{M}_1^4 \longrightarrow \mathbb{C}^* \setminus \{\pm 1\}$$

where $\overline{M}_{1,\lambda}^4 = \left\{(A,B,C) \mid [A,B] = -C^2 = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}\right\}$. If we write σ for the map $\sigma : \mathbb{C}^* \to \mathbb{C}^*$ that takes λ to $-\lambda$, noting that $R(Z/\mathbb{Z}_2) = T + S_2$, we obtain that

$$R(\sigma^*(Z/\mathbb{Z}_2)) = \sigma^*(R(Z/\mathbb{Z}_2)) = \sigma^*(T + S_2) = T + S_{-2}$$

since σ and the $\mathbb{Z}_2$-action commute. So

$$\begin{aligned} R(\overline{M}_1^4/\mathbb{Z}_2) &= R(\overline{X}_4/\mathbb{Z}_2) \otimes R(\sigma^* Z/\mathbb{Z}_2) \\ &= (q^3 T - 3qS_2 + 3q^2 S_{-2} - S_0) \otimes (T + S_{-2}) \\ &= (q^3 + 3q^2)T + (-3q - 1)S_2 + (q^3 + 3q^2)S_{-2} + (-3q - 1)S_0. \end{aligned}$$

If we write $R(\overline{M}_1^4/\mathbb{Z}_2) = a'T + b'S_2 + c'S_{-2} + d'S_0$, using Proposition 2.4.3

$$\begin{aligned} e(M_1^4) &= q(q-1)e(\overline{M}_1^4/\mathbb{Z}_2) + qe(\overline{M}_1^4) \\ &= q(q^2 - 2q - 1)a' - q(q+1)(b' + c') - 2qd' \\ &= q^6 - 11q^4 - 3q^3 + 10q^2 + 3q. \end{aligned}$$

Computation of $\mathcal{M}_{-\mathrm{Id}}$

Since not all orbits are closed, we need to distinguish between reducible and irreducible orbits in order to study the GIT quotient

$$\mathcal{M}_{-\mathrm{Id}}(\Sigma) := M_1//SL(2,\mathbb{C}).$$

All reducible representations lie in M_1^0, since a necessary condition for the pair (A,B) to be reducible is that $\mathrm{tr}([A,B]) = 2$.

Irreducible representations in M_1^0

We consider the following two cases:

- $(t_1,t_2)\neq(\pm 2,\pm 2)$. Since $[A,B]=\mathrm{Id}$, both A and B are simultaneously diagonalizable and they are characterized by their eigenvalues $(\lambda,\mu)\in\mathbb{C}^*\times\mathbb{C}^*-\{(\pm 1,\pm 1)\}$ and eigenvectors $\{e_1,e_2\}\in\mathbb{P}^1$. All elements in $C\in C_1\sqcup C_2$ are conjugate because $\operatorname{tr}C=0$, $\det C=1$; a consequence is that C is characterized by the eigenvalues $\{\pm i,\mp i\}$ and the eigenvectors $\{e_3,e_4\}\in\mathbb{P}^1$. The invariant under conjugation for the points $\{e_1,e_2,e_3,e_4\}$ is the cross ratio $r\in\mathbb{P}^1-\{0,1\}$. The stratum is parametrized by elements $(\lambda,\mu,\pm i,\mp i,r)$ under the equivalence relation

$$(\lambda,\mu,\pm i,\mp i,r)\sim(\lambda^{-1},\mu^{-1},\pm i,\mp i,1-r)\sim(\lambda,\mu,\mp i,\pm i,1-r)$$

  If we write $V_1:=\mathbb{C}^*\times\mathbb{C}^*-\{(\pm 1,\pm 1)\}$, $V_2=\{\pm(i,-i)\}$ and $V_3=\mathbb{P}-\{0,1\}$ and actions $\tau_1((\lambda,\mu))=(\lambda^{-1},\mu^{-1})$, $\tau_2(\pm(i,-i))=\mp(i,-i)$ and $\sigma(r)=1-r$ on V_1,V_2,V_3 respectively, we are looking for $e(I_1)=e(V_1\times V_2\times V_3)^{\tau_1\times\sigma,\tau_2\times\sigma}$. Therefore, since under the actions $e(V_1)^+=q^2-3, e(V_1)^-=-2q$, $e(V_2)^+=e(V_2)^+=1$ and $e(V_3)^+=q-1, e(V_3)^-=-1$,

$$\begin{aligned}e(V_1\times V_2\times V_3)^{\tau_1\times\sigma,\tau_2\times\sigma}&=e(V_1)^+e(V_2)^+e(V_3)^++e(V_1)^-e(V_2)^-e(V_3)^-\\&=(q^2-3)(q-1)+(-2q)(-1)\\&=q^3-q^2-q+3\end{aligned}$$

- $(t_1,t_2)\pm(\pm 2,\pm 2)$. In this case either A or B is of Jordan type, since otherwise the representation would be reducible. Let us assume first that A is of Jordan type. The pair (A,B) shares an eigenvector $\{f\}$ and C is characterized by two eigenvectors $\{e_1,e_2\}$ associated to $\{i,-i\}$ respectively. With respect to the basis $\{f,e_1\}$, if we reescale f we obtain that

$$A=\begin{pmatrix}1&1\\0&1\end{pmatrix}\quad B=\begin{pmatrix}1&x\\0&1\end{pmatrix}\quad C=\begin{pmatrix}-i&0\\y&i\end{pmatrix}$$

  parametrized by $(x,y)\in\mathbb{C}\times\mathbb{C}^*$. If A is not of Jordan type, $A=\mathrm{Id}$ and B is. In this case, similar computations yield that in a suitable basis, $A=\mathrm{Id}$, $B=J_+$ and C depends on a parameter $y\in\mathbb{C}^*$. Therefore

$$e(I_2)=q(q-1)+(q-1)=q^2-1.$$

Reducible representations in M_1^0

A reducible representation $(A,B,C)\in M_1^0$ is S-equivalent to the triple

$$A=\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}\quad B=\begin{pmatrix}\mu & 0\\ 0 & \mu^{-1}\end{pmatrix}\quad C=\begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix},$$

They are parametrized by elements $(\lambda,\mu,\pm i)\in\mathbb{C}^*\times\mathbb{C}^*\times\{\pm i\}$ under the equivalence relation $(\lambda,\mu,i)\sim(\lambda^{-1},\mu^{-1},-i)$ given by the permutation of the eigenvectors. Hence

$$e(R)=(q-1)^2.$$

All representations in M_1^1, M_1^3, M_1^4 are irreducible, so we obtain

$$e(I_3)=\frac{e(M_1^1)+e(M_1^3)+e(M_1^4)}{e(\mathrm{PGL}(2,\mathbb{C}))}=q^3+2q^2-4q-1.$$

Finally,

$$e(\mathcal{M}_{-\mathrm{Id}}(\Sigma))=e(I_1)+e(I_2)+e(I_3)+e(R)=2q^3+3q^2-7q+2.$$

3.7 E-polynomial of the character variety of Σ with diagonalizable holonomy

We consider the space

$$\overline{M}_4=\{(A,B,C)\mid [A,B]=\xi_{\lambda_0}C^2\},$$

where $\xi_{\lambda_0}=\begin{pmatrix}\lambda_0 & 0\\ 0 & \lambda_0^{-1}\end{pmatrix}$, $\lambda_0\neq 0,\pm1$. Let us write

$$\nu:=C^2=\begin{pmatrix}a & b\\ c & d\end{pmatrix},$$

so that

$$\delta:=\xi_{\lambda_0}C^2=\begin{pmatrix}\lambda_0 a & \lambda_0 b\\ \lambda_0^{-1}c & \lambda_0^{-1}d\end{pmatrix}.$$

Let $t_1=\operatorname{tr}\nu=a+d$ and $t_2=\operatorname{tr}\delta=\lambda_0 a+\lambda_0^{-1}d$. Note that (t_1,t_2) determine (a,d), since

$$a=\frac{-\lambda_0^{-1}t_1+t_2}{\lambda_0-\lambda_0^{-1}},\qquad d=\frac{\lambda_0 t_1-t_2}{\lambda_0-\lambda_0^{-1}}.$$

We need to distinguish when these two matrices have a common eigenvector, i.e. when $ad=1$. In terms of the traces, this condition can be rewritten as a conic in the (t_1,t_2)-plane,

$$H:=\{(t_1,t_2)\mid -t_1^2-t_2^2+(\lambda_0+\lambda_0^{-1})t_1t_2=(\lambda_0-\lambda_0)^2\}.$$

We get:

Proposition 3.7.1. *ν, δ share an eigenvector if and only if $(t_1, t_2) \in H$*

Proof. One implication is clear, we focus on the if part. The matrices ν, δ share an eigenvector $v = (x, y)$ if and only if v is a common point of the two conics C_1, C_2 given by:

$$\begin{cases} cx^2 + (d-a)xy - by^2 & = 0 \\ cx^2 + (d - \lambda_0^2 a)xy - \lambda_0^2 by^2 & = 0 \end{cases}$$

But both conics share a common point if and only if their resultant is equal to zero. A simple computation shows that $Res(C_1, C_2) = -bc \det \nu$, which vanishes if and only if $bc = 0$. □

We stratify $\overline{M}_4$ according to the map given by the traces

$$\overline{M}_4 \mapsto \mathbb{C}^2$$
$$(A, B, C) \mapsto (\operatorname{tr} C^2, \operatorname{tr}[A, B]),$$

taking into account the special cases when $t_i = \pm 2$ and $(t_1, t_2) \in H$.

Points: $t_1, t_2 = \pm 2$

- $t_1 = 2$. In this case, if $ad = 1$ then $a = d = 1$ and therefore $t_2 = \pm 2 = \pm\lambda_0 + \lambda_0^{-1}$, a contradiction. So $C^2 \sim J_+$ with $bc \neq 0$. For every such $b \in \mathbb{C}^*$, c is determined by $\det C^2 = 1$. There are two square roots C in this positive Jordan case, considering the two cases $t_2 = \pm 2$ we get

$$e(M_4^1) = 2(q-1)(e(\overline{X}_2) + e(\overline{X}_3)) = 4q^4 - 2q^3 - 8q^2 + 6q.$$

- $t_1 = -2$. Again, it is necessary that $bc \neq 0$, so C^2 is of negative Jordan type. However, there are no solutions to the equation $C^2 \sim J_-$ in $SL(2, \mathbb{C})$ and the stratum is empty.

Intersections of H and the lines $t_i = \pm 2$

$t_1 = 2, t_2 = \lambda_0 + \lambda_0^{-1}$

This stratum is the intersection of the hyperbola H (condition $ad = 1$) and the line of equation $t_1 = 2$. In this situation $bc = 0$, so we get three possible cases:

- $b = c = 0$. In this case $C^2 = \mathrm{Id}$, so $C = \pm\,\mathrm{Id}$, $[A, B] = \xi_{\lambda_0}$ and

$$e(M_4^{2,1}) = 2e(\overline{X}_{4,\lambda_0}) = 2q^3 + 6q^2 - 6q - 2.$$

- $b \neq 0, c = 0$ and $b = 0, c \neq 0$. We compute the first case, the other one being similar. For every $b \in \mathbb{C}^*$, C^2 is of positive Jordan type and $[A, B] = \xi_{\lambda_0} C^2 \sim \xi_{\lambda_0}$. Doubling the contribution to take into account both cases,

$$e(M_4^{2,2}) = 2((q-1)2e(\overline{X}_{4,\lambda_0})) = 4q^4 + 8q^3 - 24q^2 + 8q + 4.$$

$t_1 = -2, t_2 = -\lambda_0 - \lambda_0^{-1}$

This case corresponds to the intersection of H and $t_1 = -2$, so $bc = 0$ and again:

- $b = c = 0$. $C^2 = -\operatorname{Id}$ implies that $C \in C_1$ or $C \in C_2$, $[A, B] = -\xi_{\lambda_0}$, so

$$e(M_4^{2,3}) = e(C_1 \sqcup C_2)e(\overline{X}_{4,-\lambda_0}) = q^5 + 4q^4 - 4q^2 - q.$$

- If $b = 0, c \neq 0$ or $b \neq 0, c = 0$, then C^2 is of negative Jordan type and the stratum is empty.

$t_1 = \lambda_0 + \lambda_0^{-1}, t_2 = 2$

In this case, the intersection of the line $t_2 = 2$ and H yields once more $bc = 0$, but note that now C^2 is diagonalizable:

- $b = c = 0$. From the equations, we deduce that $a = \lambda_0^{-1}$ and $d = \lambda_0$. The equation $C^2 = \begin{pmatrix} \lambda_0^{-1} & 0 \\ 0 & \lambda_0 \end{pmatrix}$ has two solutions and $[A, B] = \xi_{\lambda_0} C^2 = \operatorname{Id}$, so $(A, B) \in X_0$. We obtain

$$e(M_4^{2,4}) = 2e(X_0) = 2q^4 + 8q^3 - 2q^2 - 8q.$$

- $b \neq 0, c = 0$, or $b = 0, c \neq 0$. In the first case, $b \in \mathbb{C}^*$ and C^2 is of diagonal type. We have $a = \lambda_0^{-1}$, $d = \lambda_0$ and $[A, B]$ is of Jordan type J_+. The case $b = 0, c \neq 0$ is similar, so doubling the E-polynomial

$$e(M_4^{2,5}) = 2(2(q-1)e(\overline{X}_2)) = 4q^4 - 12q^3 - 4q^2 + 12q.$$

$t_1 = -\lambda_0 - \lambda_0^{-1}, t_2 = -2$

C^2 is of diagonal type, and $a = -\lambda_0^{-1}$, $d = -\lambda_0$. Similar computations yield

- $b = c = 0, \quad e(M_4^{2,6}) = 2e(X_1) = 2q^3 - 2q.$

- $b = 0, c \neq 0,\ b \neq 0, c = 0 \quad e(M_4^{2,7}) = 4(q-1)e(\overline{X}_3) = 4q^4 + 8q^3 - 12q^2.$

Adding up, we obtain

$$e(M_4^2) = \sum_{i=1}^{7} e(M_4^{2,i}) = q^5 + 18q^4 + 16q^3 - 40q^2 + 3q + 2.$$

Lines $t_i = \pm 2$

We compute now the remaining points in each of the lines $t_i = \pm 2$.

$t_1 = 2$**,** $t_2 \neq \pm 2, \lambda_0 + \lambda_0^{-1}$

First of all, notice that C^2 must be of positive Jordan type since otherwise we would have that $t_2 = \lambda_0 + \lambda_0^{-1}$, a special point we have already considered. Also $(t_1, t_2) \notin H$ implies that $ad \neq 1$. We are dealing with the line $L = \{(2, t_2), t_2 \neq \pm 2, \lambda_0 + \lambda_0^{-1}\}$ in $\mathbb{C}^2$. If we fix $b = 1$ conjugating by an element of $\mathrm{Stab}\,(\xi_0) = D$, then

$$C^2 = \begin{pmatrix} a & 1 \\ ad - 1 & d \end{pmatrix}$$

is of Jordan type J_+. We can conjugate again by $\begin{pmatrix} 1 & 0 \\ a - 1 & 1 \end{pmatrix}$ and obtain $C^2 = J_+$ for all $t_2 \in L$, so that the family gets trivialized over the line. To deal with the remaining part of the fibration, for every $t_2 \in L$ we may take the double cover given by $\mu \mapsto t_2 := \mu + \mu^{-1}$, and quotient in a second step by the $\mathbb{Z}_2$-action that takes μ to μ^{-1}. This has E-polynomial $e(\overline{X}_4/\mathbb{Z}_2)$. We have to remove the fibre corresponding to $t_2 = \lambda_0 + \lambda_0^{-1}$, so:

$$e(M_4^{3,1}) = 2(q-1)(e(\overline{X}_4/\mathbb{Z}_2) - e(\overline{X}_{4,\lambda_0})) = 2q^5 - 8q^4 - 6q^3 + 24q^2 - 8q - 4.$$

$t_1 = -2, t_2 \neq \pm 2, \lambda_0 + \lambda_0^{-1}$

In this case, $C^2 \sim J_-$ (it cannot be equal to $-\mathrm{Id}$, since $bc \neq 0$). The stratum is empty.

$t_1 \neq \pm 2, \lambda_0 + \lambda_0^{-1}, t_2 = 2$

In this case, C^2 is diagonalizable with eigenvalues different from $\{\lambda_0, \lambda_0^{-1}\}$. Because of it, $[A, B]$ must be of Jordan type J_+. On one hand, as we did before, we can trivialize this family over L so that $[A, B] = J_+$. On the other hand, the fibration given by

$$C^2 = \begin{pmatrix} a & 1 \\ ad - 1 & d \end{pmatrix},$$

where $t_1 = a + d \in L$, can be solved taking a double cover as before, obtaining that that part of the fibration is isomorphic to $(\overline{Z}/\mathbb{Z}_2) \setminus Z_{\lambda_0}$. Hence

$$\begin{aligned} e(M_4^{3,2}) &= (q-1)e(\overline{X}_2)e((\overline{Z}/\mathbb{Z}_2) \setminus Z_{\lambda_0}) \\ &= (q-1)(q^3 - 2q^2 - 3q)(q-5) \\ &= q^5 - 8q^4 + 14q^3 + 8q^2 - 15q. \end{aligned}$$

$t_1 \neq \pm 2, \lambda_0 + \lambda_0^{-1}, t_2 = -2$

Similarly,

$$e(M_4^{3,3}) = (q-1)e(\overline{X}_3)e((\overline{Z}/\mathbb{Z}_2) \setminus Z_{\lambda_0}) = q^5 - 3q^4 - 13q^3 + 15q^2.$$

Adding all up, the contribution given by the lines is

$$e(M_4^3) = \sum_{i=1}^{3} e(M_2^{3,i}) = 4q^5 - 19q^4 - 5q^3 + 47q^2 - 23q - 4.$$

Hyperbola H

We move to the case: $(t_1, t_2) \in H$, $t_i \neq \pm 2$. First of all, since $(t_1, t_2) \in H$, $bc = 0$. If we write $t_1 = \mu + \mu^{-1}$ note that this eigenvalue μ can be used to parametrize H via the map

$$\begin{aligned} \mathbb{C}^* \setminus \{\pm 1, \pm\lambda_0\} &\mapsto H \\ \mu &\mapsto (t_1, t_2) = (\mu + \mu^{-1}, \lambda_0\mu + \lambda_0^{-1}\mu^{-1}). \end{aligned}$$

We get a fibration $\overline{E}$ over the punctured line $B = \mathbb{C}^* \setminus \{\pm 1, \pm\lambda_0\}$ defined by the parameter. For every μ, both C^2 and $[A, B]$ are of diagonal type (with eigenvalues $\{\mu, \mu^{-1}\}$ and $\{\lambda_0\mu, \lambda_0^{-1}\mu^{-1}\}$ respectively) so the fibre is isomorphic to $\overline{X}_{4,\mu} \times \overline{Z}_{\lambda_0\mu}$.

If we write $\sigma' : \mathbb{C}^* \mapsto \mathbb{C}^*$ for the map which takes $\mu \mapsto \lambda_0\mu$, then $\overline{E} \cong \overline{X}_4 \times_B \sigma'^* \overline{Z}$. Using the monodromy representations

$$\begin{aligned} R(\overline{E}) &= R(\overline{X}_4) \otimes \sigma'^*(R(Z)) \\ &= ((q^3 - 1)T + (3q^2 - 3q)N) \otimes (T + N) \\ &= (q^3 + 3q^2 - 3q - 1)T + (q^3 + 3q^2 - 3q - 1)N. \end{aligned}$$

If we write $R(\overline{E}) = a''T + b''N$, using Proposition 2.3.6 and multiplying by $(2q - 1)$ to account for the three possibilities for $bc = 0$

$$\begin{aligned} e(M_4^4) &= (2q - 1)e(\overline{E}) \\ &= (2q - 1)((q - 5)a'' - 4b'') \\ &= 2q^5 - 13q^4 - 54q^3 + 82q^2 - 8q - 9. \end{aligned}$$

General case

Finally, we take into account the general case: $(t_1, t_2) \notin H$, $t_i \neq \pm 2$. Now $bc \neq 0$ and fixing $b \in \mathbb{C}^*$, a, b, c, d are all determined by every pair of traces (t_1, t_2). The fibre over each (t_1, t_2) is isomorphic to $\overline{X}_{4,\mu_1} \times \overline{Z}_{\mu_2}$, where $t_i = \mu_i + \mu_i^{-1}, i = 1, 2$.

If we forget about the condition $(t_1, t_2) \notin H$ (we can substract later its contribution, $e(\overline{E})$ above), then the total E-polynomial is $e(\overline{X}_4/\mathbb{Z}_2)e(\overline{Z}/\mathbb{Z}_2)$. Therefore

$$\begin{aligned} e(M_4^5) &= (q - 1)(e(\overline{X}_4/\mathbb{Z}_2)e(Z/\mathbb{Z}_2) - e(\overline{E})) \\ &= q^6 - 7q^5 + 15q^4 + 33q^3 - 76q^2 + 22q + 12. \end{aligned}$$

Reducible orbits

Because of Proposition 3.7.1, $[A,B]$ and C^2 share an eigenvector if and only if $(t_1,t_2)\in H$. Moreover, (A,B) is reducible if and only if $\mathrm{tr}([A,B])=2$, so the set of reducible orbits lies in $H\cap\{t_2=2\}$, which corresponds to the strata $M_4^{2,4}$ and $M_4^{2,5}$. We describe the reducible locus in each case and identify the orbits that are S-equivalent.

- $M_4^{2,4}(b=c=0)$. In this case $C^2=\xi_{\lambda_0}$ and $[A,B]=\mathrm{Id}$, so a representation given by (A,B,C) will be reducible if and only if the pair $(A,B)\in X_0$ is upper or lower-triangular. We will double each contribution to take into account the two possible values for C. Looking at the description of X_0 given in [53], we have the following options for (A,B),

  - $A=\pm\mathrm{Id}, B\in SL(2,\mathbb{C})$ and $B=\pm\mathrm{Id}, A\in SL(2,\mathbb{C})$. In either case, the upper and lower triangular matrices give a contribution of $(2q-1)(q-1)$. We need to substract $(\pm\mathrm{Id},\pm\mathrm{Id})$, so the reducible locus in this case has E-polynomial

$$\begin{aligned} e(\mathcal{R}_1) &= 2(4(2q-1)(q-1)-4)=16q^2-24q. \\ e(\mathcal{I}_1) &= 2(2e(SL(2,\mathbb{C})\times\{\pm\mathrm{Id}\})-e(\{(\pm\mathrm{Id},\pm\mathrm{Id})\}))-e(R_1) \\ &= 8q^3-16q^2+16q-8. \end{aligned}$$

  - (A,B) are simultaneously diagonalizable. If we write $\Omega=\mathbb{C}^*\times\mathbb{C}^*-\{(\pm1,\pm1)\}$, this stratum is isomorphic to $(\Omega\times GL(2,\mathbb{C})/D)/\mathbb{Z}_2$, where the $\mathbb{Z}_2$-action is given by the interchange of the eigenvalues. The set of upper-triangular matrices is given by those matrices in $GL(2,\mathbb{C})/D$ that leave the pair

$$\left(\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix},\begin{pmatrix}\mu & 0\\ 0 & \mu^{-1}\end{pmatrix}\right) \tag{3.36}$$

    upper-triangular, which we will denote by $H\subset GL(2,\mathbb{C})/D$. Using the isomorphism $GL(2,\mathbb{C})/D\cong\mathbb{P}^1\times\mathbb{P}^1\setminus\Delta$ that takes $\left(\begin{smallmatrix}x & y\\ z & t\end{smallmatrix}\right)$ to $([x,y],[z,t])$, a simple computation shows that the set of matrices in $GL(2,\mathbb{C})$ that leave the pair upper-triangular are

$$H_1=\left\{\left[\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix}\right]\right\},\quad H_2=\left\{\left[\begin{pmatrix}0 & 1\\ 1 & t\end{pmatrix}\right]\right\},$$

    where $y,t\in\mathbb{C}$. Note that the action of $\mathbb{Z}_2$ interchanges H_1 and H_2. Therefore $e(\mathbb{C}^*\times\mathbb{C}^*-\{(\pm1,\pm1)\})^+=q^2-3$, $e(\mathbb{C}^*\times\mathbb{C}^*-\{(\pm1,\pm1)\})^-=-2q$, $e(H)^+=e(H_1)=q$ and $e(H)^-=e(H_2)=q$. The computation of the elements that leave the pair strictly lower-triangular is analogous, but in that case the parameters $y, t \in \mathbb{C}^*$. We obtain

$$\begin{aligned} e(R_2) &= 2((q^2-3)q + (-2q)q + (q^2-3)(q-1) + (-2q)(q-1)) \\ &= 4q^3 - 10q^2 - 8q + 6, \\ e(I_2) &= 2(e(\Omega \times GL(2,\mathbb{C})/D)/\mathbb{Z}_2)) - e(R_2) = 2q^4 - 4q^3 + 8q - 6. \end{aligned}$$

– (A, B) are of Jordan type. Let us assume that $(\operatorname{tr} A, \operatorname{tr} B) = (2, 2)$, the other three remaining cases are similar. After a choice of basis and conjugating by a suitable element in U, we may assume that

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}, \tag{3.37}$$

where $y \in \mathbb{C}^*$. For each $y \in \mathbb{C}^*$ the orbit is isomorphic to $GL(2,\mathbb{C})/U$, so the reducible locus will be given by those $H' \subset GL(2,\mathbb{C})/U$ that leave the pair (A, B) upper or lower-triangular. Given $Z = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, the action of an element $\left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right) \in U$ by left multiplication gives us the following choices of representatives inside its equivalence class in $GL(2,\mathbb{C})/U$:

* $Z_1 := \{Z \in GL(2,\mathbb{C})/U \mid \gamma \neq 0\}$. Choosing $x = 1/\gamma, y = -\alpha/\gamma^2$, $Z \sim \begin{pmatrix} 0 & \beta' \\ 1 & \delta' \end{pmatrix}$, $\beta' \in \mathbb{C}^*, \delta' \in \mathbb{C}$.
* $Z_2 := \{Z \in GL(2,\mathbb{C})/U \mid \gamma = 0\}$. Taking $x = 1/\delta, -\beta/\delta^2$, $Z \sim \begin{pmatrix} \alpha' & 0 \\ 0 & 1 \end{pmatrix}$, $\alpha' \in \mathbb{C}^*$.

The only matrices in Z_1 that leave the pair 3.37 lower-triangular are the ones such that $\delta' = 0$, while none of them leave it upper-triangular. All elements in Z_2 leave 3.37 upper-triangular. If we write H' for this subgroup of $GL(2,\mathbb{C})$, we see that $e(H') = (q-1) + (q-1) = 2(q-1)$. Therefore

$$\begin{aligned} e(R_3) &= 8e(\mathbb{C}^*)e(H) = 16(q-1)^2 \\ e(I_3) &= 8e(\mathbb{C}^*)e(GL/U) - e(R_3) = 8q^3 - 24q^2 + 24q - 8. \end{aligned}$$

We obtain that the E-polynomial of the set of irreducible orbits is

$$e(M_4^{2,4,irr}) = e(I_1) + e(I_2) + e(I_3) = 2q^4 + 12q^3 - 40q^2 + 48q - 22.$$

For the reducible locus, every representation (A, B, C) is S-equivalent to a triple of the form:

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad B = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}, \quad C = \begin{pmatrix} \pm\sqrt{\lambda_0} & 0 \\ 0 & \pm\sqrt{\lambda_0^{-1}} \end{pmatrix} \tag{3.38}$$

where $(\lambda,\mu)\in(\mathbb{C}^*)^2$, under the equivalence relation $(\lambda,\mu,\pm\sqrt{\lambda_0})\sim(\lambda^{-1},\mu^{-1},\pm\sqrt{\lambda_0^{-1}})$. The space is parametrized by $(\mathbb{C}^*)^2\times\{\pm\sqrt{\lambda_0}\}$, so

$$e(M_4^{2,4,red})=2(q-1)^2$$

- $b\neq 0, c=0$ and $b\neq 0, c=0$. In the first case, $[A,B]$ is of Jordan type J_+ and C^2 is of diagonal type. All representations in this case are reducible, and using the action of D, they are S-equivalent to the reducible orbits given in (3.38). The second case is analogous.

Adding up all the irreducible orbits

$$e(M_4^{irr})=\sum_{i=1}^{5}e(M_4^i)=q^6+q^4+4q^3-29q^2+44q-21,$$

and also

$$e(M_4^{red})=e(M_4^{2,5,red})=2(q-1)^2.$$

Finally,

$$e(\mathcal{M}_{\xi_{\lambda_0}}(\Sigma))=e(M_4^{irr})/(q-1)+e(M_4^{red})=q^5+q^4+2q^3+8q^2-27q+23.$$

3.8 E-polynomial of the $SL(2,\mathbb{C})$-character variety of Σ of Jordan type J_+

Let us consider

$$M_2=\tilde{g}^{-1}(J_+)=\{(A,B,C)\in SL(2,\mathbb{C})^3\mid[A,B]=J_+C^2\}$$

and the associated moduli space

$$\mathcal{M}_{J_+}(\Sigma)=M_2/\operatorname{Stab}(J_+).$$

As we previously did in Section 3.7, we introduce equations for $[A,B]$ and C^2. Let us write

$$C^2=\begin{pmatrix}a&b\\c&d\end{pmatrix}$$

so that

$$[A,B]=J_+C^2=\begin{pmatrix}a+c&b+d\\c&d\end{pmatrix}.$$

We stratify M_2 according to the different values for $t_1=\operatorname{tr}C^2=a+d$ and $t_2=\operatorname{tr}[A,B]=a+d+c$. Note that $c=t_2-t_1$.

Points $(t_1, t_2) = (\pm 2, \pm 2)$

- $(2,2)$. In this case, $c = 0$, so $a = d = 1$. If $b \neq 0, -1$, then both C^2 and $[A,B]$ are of Jordan type J_+, giving a contribution of $2(q-2)e(\overline{X}_2)$. If $b = 0$, then $C^2 = \mathrm{Id}$ and $[A,B]$ is of Jordan type, which gives us $2e(\overline{X}_2)$. If $b = -1$, then $[A,B] = \mathrm{Id}$ and C^2 is of Jordan type J_+, which adds $2e(X_0)$. Adding all up

$$e(M_2^{1,1}) = (q-2)2e(\overline{X}_2) + 2e(\overline{X}_2) + 2(X_0) = 4q^4 + 2q^3 - 4q^2 - 2q.$$

- $(-2,-2)$. Again, $c = 0$ and $a = d = -1$. C^2 is not of Jordan type J_- if and only if $b = 0$, in which case $C^2 = -\mathrm{Id}$ and $[A,B]$ is of Jordan type J_-. Therefore:

$$e(M_2^{1,2}) = (e(C_1) + e(C_2))e(\overline{X}_3) = q^5 + 4q^4 + 3q^3.$$

- $(2,-2)$. In this case $c \neq 0$ and conjugating by an element in $\mathrm{Stab}\,(J_+)$, we can assume that $d = 0$, so $b = -\frac{1}{c}$ and $a = t_1$. Therefore a, b, c, d are fixed and C^2 is of Jordan type J_+. $[A,B]$ is of Jordan type J_-, so

$$e(M_2^{1,3}) = 2qe(\overline{X}_3) = 2q^4 + 6q^3.$$

- $(-2,2)$. The same computations as in the previous case give us that C^2 is of Jordan type J_-, so the stratum is empty.

Hence

$$e(M_2^1) = e(M_2^{1,1}) + e(M_2^{1,2}) + e(M_2^{1,3}) = q^5 + 10q^4 + 11q^3 - 4q^2 - 2q.$$

Lines $\{t_i = \pm 2\}$**,** $\{t_1 = t_2\}$

$t_1 = 2, t_2 \neq \pm 2$

In this stratum $c = t_2 - 2 \neq 0$, so C^2 is of positive Jordan type. Conjugating by an element in $\mathrm{Stab}\,(J_+)$ we can assume that $d = 0$, $b = -\frac{1}{c}$ and $a = t_1 = 2$, giving that

$$C^2 = \begin{pmatrix} 2 & \frac{1}{2-t_2} \\ t_2 - 2 & 0 \end{pmatrix}, \quad [A,B] = \begin{pmatrix} t_2 & \frac{1}{2-t_2} \\ t_2 - 2 & 0 \end{pmatrix},$$

which defines a fibration over the line $L \cong \mathbb{C} \setminus \{\pm 2\}$ parametrized by t_2. Conjugating by a suitable matrix C^2 can be put into Jordan form J_+ and the fibration given by such C is trivial. To deal with the fibration given by the pairs (A,B), we can consider the double cover $\mu \mapsto t_2 = \mu + \mu^{-1}, \mu \neq 0, \pm 1$, and quotient later by the $\mathbb{Z}_2$-action given by $\mu \mapsto \mu^{-1}$. The total space is isomorphic to $\overline{X}_4/\mathbb{Z}_2$, which means that this subcase is isomorphic to $\mathbb{C} \times \{\pm\sqrt{J_+}\} \times \overline{X}_4/\mathbb{Z}_2$. Therefore

$$e(M_2^{2,1}) = 2qe(\overline{X}_4/\mathbb{Z}_2) = 2q^5 - 4q^4 - 6q^3 + 6q^2 + 2q.$$

$t_1 = -2, t_2 \neq \pm 2$

In this case $c \neq 0$ implies that C^2 is of Jordan type J_-, which does not have solutions in $SL(2, \mathbb{C})$. The stratum is empty.

$t_1 \neq \pm 2, t_2 = 2$

Since $c = 2 - t_1 \neq 0$, we see that $[A, B]$ is of Jordan type J_+ and C^2 is diagonalizable. After conjugating by an element in $\mathrm{Stab}\,(J_+)$, we can assume that $d = 0$, $a = t_1$ and $b = -\frac{1}{c}$;

$$C^2 = \begin{pmatrix} t_1 & \frac{1}{t_1 - 2} \\ 2 - t_1 & 0 \end{pmatrix}, \quad [A, B] = \begin{pmatrix} 2 & \frac{1}{t_1 - 2} \\ 2 - t_1 & 0 \end{pmatrix}$$

which defines a fibration over the punctured line $L = \{(t_1, 2), t_1 \neq \pm 2\}$. Like we did before, trivializing one part of the fibration by conjugation and working with a double cover on remaining one, we obtain the contribution

$$e(M_2^{2,2}) = qe(\overline{X}_2)e(\overline{Z}/\mathbb{Z}_2) = q(q^3 - 2q^2 - 3q)(q - 3) = q^5 - 5q^4 + 3q^3 + 9q^2.$$

$t_1 \neq \pm 2, t_2 = -2$

This case is analogous to the previous one. We obtain

$$e(M_2^{2,3}) = qe(\overline{X}_3)e(\overline{Z}/\mathbb{Z}_2) = q^5 - 9q^3.$$

$t_1 = t_2 \neq \pm 2$

Now $c = t_2 - t_1 = 0$, so $d = a^{-1}$ and we get a $\mathbb{C}^*$-parameter given by $b \in \mathbb{C}^*$. We have a fibration over the line $\{t_1 \in \mathbb{C} \setminus \{\pm 2\}\}$, $t_1 = t_2 = a + a^{-1}$, with fibre over (t_1, t_1)

$$C^2 = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, \quad [A, B] = \begin{pmatrix} a & b + a^{-1} \\ 0 & a^{-1} \end{pmatrix},$$

isomorphic to the disjoint union of $\overline{Z}_a \times \overline{X}_{4,a}$ and $\overline{Z}_{a^{-1}} \times \overline{X}_{4,a^{-1}}$. If we lift the fibration to the double cover given by $a \in \mathbb{C} \setminus \{0, \pm 1\}$, where $a \mapsto t_1 = a + a^{-1}$, we get a fibration $\tilde{E}$ over $\mathbb{C} \setminus \{0, \pm 1\}$, with fibre isomorphic to $\overline{Z}_a \times \overline{X}_{4,a}$, isomorphic to the previous one. To obtain its E-polynomial we compute its Hodge monodromy representation

$$\begin{aligned} R(\tilde{E}) &= R(\overline{X}_4) \otimes R(\overline{Z}) \\ &= ((q^3 - 1)T + (3q^2 - 3q)N) \otimes (T + N) \\ &= (q^3 + 3q^2 - 3q - 1)T + (q^3 + 3q^2 - 3q - 1)N. \end{aligned}$$

Using Proposition 2.4.3, we get

$$\begin{aligned} e(\tilde{E}) &= (q - 3)(q^3 + 3q^2 - 3q - 1) - 2(q^3 + 3q^2 - 3q - 1) \\ &= q^4 - 2q^3 - 18q^2 + 14q + 5, \end{aligned}$$

and therefore

$$e(M_2^{2,4}) = qe(\tilde{E}) = q^5 - 2q^4 - 18q^3 + 14q^2 + 5q.$$

The total sum of the E-polynomials given by these lines is

$$e(M_2^2) = \sum_{i=1}^{4} e(M_2^{2,i}) = 5q^5 - 11q^4 - 30q^3 + 29q^2 + 7q.$$

General case

We consider now the open subset of $(\mathbb{C}^*)^2$ given by $t_1 \neq t_2$, $t_1, t_2 \neq \pm 2$. Since $c = t_2 - t_1 \neq 0$, after conjugation we can assume that $d = 0$, $b = -\frac{1}{c}$ and $a = t_1$. The fibre over each (t_1, t_2) is the set of $(A, B, C) \in SL(2, \mathbb{C})^3$ such that:

$$C^2 = \begin{pmatrix} t_1 & -\frac{1}{t_2 - t_1} \\ t_2 - t_1 & 0 \end{pmatrix}, \quad [A, B] = \begin{pmatrix} t_2 & -\frac{1}{t_2 - t_1} \\ t_2 - t_1 & 0 \end{pmatrix},$$

isomorphic to $\mathbb{C} \times \overline{Z}_{\lambda_1} \times \overline{X}_{4,\lambda_2}$, where $t_i = \lambda_i + \lambda_i^{-1}$. Ignoring the condition $t_1 \neq t_2$, the total space is isomorphic to $\mathbb{C} \times \overline{Z}/\mathbb{Z}_2 \times \overline{X}_4/\mathbb{Z}_2$. It remains to substract the contribution over the line $(t_1, t_1), t_1 \neq \pm 2$, with fibre $\mathbb{C} \times \overline{Z}_\lambda \times \overline{X}_{4,\lambda}$. It is isomorphic to $\mathbb{C} \times \overline{M}_0^4/\mathbb{Z}_2$, and its E-polynomial was computed in Section 3.5, so

$$\begin{aligned} e(M_2^3) &= q(e(Z/\mathbb{Z}_2)e(\overline{X}_4/\mathbb{Z}_2) - e(\overline{M}_0^4/\mathbb{Z}_2)) \\ &= q((q-3)(q^4 - 2q^3 - 3q^2 + 3q + 1) - (q^4 - 3q^3 - 9q^2 + 9q + 2)) \\ &= q^6 - 6q^5 + 6q^4 + 21q^3 - 17q^2 - 5q. \end{aligned}$$

Reducible orbits

Proposition 3.8.1. *$[A, B], C^2$ share an eigenvector if and only if $c = 0$*

Proof. The only if part is clear. Now, given $[A, B] = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $C^2 = \left(\begin{smallmatrix} a+c & b+d \\ c & d \end{smallmatrix}\right)$, both matrices share an eigenvector $v = (x, y)$ if and only if the conics

$$\begin{cases} cx^2 + (d-a)xy - by^2 = 0 \\ cx^2 + (d-a-c)xy - (b+d)y^2 = 0 \end{cases}$$

have a common solution, which happens only when $c = 0$, since $Res(C_1, C_2) = c^2$. $\square$

Since (A, B) is reducible if and only if $tr[A, B] = 2$, the set of reducible orbits lies entirely in the line $(t_1, 2) \cap \{c = 0\}$, which is precisely the case when $(t_1, t_2) = (2, 2)$, i.e. $M_2^{1,1}$. Note that $a = d = 1$. According to the possible values for b, we study the reducible locus and the action of U in each case:

- $b \neq 0, -1$. In this case, both C^2 and $[A, B]$ are of Jordan type. Every orbit is reducible and the action of U is geometric, it acts trivially on C and on the pair (A, B) by translations. Therefore
$$e(S_1) = 2(q-2)e(\overline{X}_2)/q = 2q^3 - 8q^2 + 2q + 12.$$

- $b = 0$. In this case $C^2 = \mathrm{Id}$, $[A, B] = J_+$. All orbits are reducible and U acts by translations. We get
$$e(S_2) = 2e(\overline{X}_2)/q = 2q^2 - 4q - 6.$$

- $b = -1$. Now $C^2 = \left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)$, so $C = \pm \begin{pmatrix} 1 & -1/2 \\ 0 & 1 \end{pmatrix}$ and $[A, B] = \mathrm{Id}$. We will double each contribution to take into account the two values of C. The reducible locus will be given by upper triangular pairs $(A, B) \in X_0$. To understand the U-action, we analyse each case separately, computing the U-action on the reducible and irreducible locus:

  - $A \in SL(2, \mathbb{C}), B = \pm\,\mathrm{Id}$ and $A = \pm\,\mathrm{Id}, B \in SL(2, \mathbb{C})$, . Given $A = \left(\begin{smallmatrix} x & y \\ z & t \end{smallmatrix}\right)$, the action of $\left(\begin{smallmatrix} 1 & \alpha \\ 0 & 1 \end{smallmatrix}\right) \in U$ takes it to $\left(\begin{smallmatrix} x-\alpha z & y-\alpha(t-x)-\alpha^2 z \\ z & t+\alpha z \end{smallmatrix}\right)$. So when $z = 0$, if $t \neq x$ we can assume that $y = 0$ and $t = x^{-1}$. If $t = x = \pm 1$, the action is trivial. Finally, if $z \neq 0$, we can arrange $t = 0$ and $y = -\frac{1}{z}$. Putting all together and substracting $e(\{(\pm\,\mathrm{Id}, \pm\,\mathrm{Id})\}) = 4$,
$$e(S_3) = 2(4((q-1) + 2q + q(q-1)) - 4) = 8q^2 + 16q - 16.$$

  - (A, B) are simultaneously diagonalizable. If we look at the action of U on $GL(2, \mathbb{C})$ by right multiplication, it takes $\left(\begin{smallmatrix} x & y \\ z & t \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} x & x\alpha+y \\ z & z\alpha+t \end{smallmatrix}\right)$. Passing to $GL(2, \mathbb{C})/D$, if $z \neq 0$, we can make $z = 1$ (by the left D-action) and $t = 0$ (by the U-action). If $z = 0$, then we can make $y = 0$, $t = x = 1$ In other words, a set of representatives for the action of U in $GL(2, \mathbb{C})/D$ is given by $\mathbb{P}^1 \times [1:0] \subset \mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta \cong GL(2, \mathbb{C})/D$. and an extra point $P := [1:0] \times [0:1]$ corresponding to $z = 0$. The $\mathbb{Z}_2$-action interchanges the rows and takes $[1:0] \times [1:y]$ to $[1:0] \times [1:-y], y \neq 0$ and takes P to $[0:1] \times [1:0]$. The eigenvalues (λ, μ) are parametrized by $(\mathbb{C}^* \setminus \{\pm 1\})^2$ modulo $(\lambda, \mu) \sim (\lambda^{-1}, \mu^{-1})$. We obtain:
$$\begin{aligned} e((\mathbb{C}^* \setminus \{\pm 1\})^2)^+ &= (q-2)^2 + 1 = q^2 - 4q + 5 \\ e((\mathbb{C}^* \setminus \{\pm 1\})^2)^- &= e(\mathbb{C}^* \setminus \{\pm 1\}) - e(\mathbb{C}^* \setminus \{\pm 1\})^+ = -2q + 4 \\ e(\mathbb{P}^1 \times [1:0] \cup P)^+ &= q + 1 \\ e(\mathbb{P}^1 \times [1:0] \cup P)^- &= 1 \end{aligned}$$
  so that
$$e(S_4) = 2((q^2 - 4q + 5)(q + 1) + (-2q + 4)) = 2q^3 - 6q^2 - 2q + 18$$

- (A, B) are of Jordan type. There are four cases, we assume that $(\operatorname{tr} A, \operatorname{tr} B) = (2, 2)$. Every such pair is equivalent to $\left(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & y \\ 0 & 1 \end{smallmatrix}\right) \right), y \in \mathbb{C}^*$, by conjugation by an element $\left(\begin{smallmatrix} x & y \\ z & t \end{smallmatrix}\right) \in GL(2, \mathbb{C})/U$ (U acts by left-multiplication as the stabilizer of the pair). We want to study the right action given by right-multiplication by U (which corresponds now to the action by conjugation of $\operatorname{Stab}(J_+)$ on M_2). Using the set of representatives for $GL(2, \mathbb{C})/U$ given in 3.37, we can use the right action to assume that $x = t = 0, z = 1$ (if $z \neq 0$) and $y = z = 0, t = 1$ (if $z = 0$, the right U-action is trivial on $GL(2, \mathbb{C})/U$ in this case). The quotient is therefore parametrized by to $\mathbb{C}^* \sqcup \mathbb{C}^*$, so

$$e(IR_5) = 16(q - 1)^2.$$

The quotient of $M_2^{1,1}$ by U has E-polynomial

$$e(M_2^{1,1}/U) = \sum_{i=1}^{5} S_i = 4q^3 + 12q^2 - 20q + 24.$$

The remaining strata consist of irreducible orbits and the action is free. If we sum the E-polynomials of these orbits

$$e(M_2) = \sum_{i=1}^{3} e(M_2^i) - e(M_2^{1,1}) = q^6 + q^4 + 12q^2 + 2q,$$

and divide by $\operatorname{Stab}(J_+) \cong U/\mathbb{C}^*$, we get

$$e(\mathcal{M}_{J_+}(\Sigma)) = e(M_2)/e(\operatorname{Stab}(J_+)) + e(M_2^{1,1}/U) = q^5 + 5q^3 + 12q^2 - 8q + 26.$$

3.9 E-polynomial of the $SL(2, \mathbb{C})$-character variety of Σ of Jordan type J_-

Let us define

$$M_3 := \tilde{g}^{-1}(J_-) = \{(A, B, C) \in SL(2, \mathbb{C})^3 \mid [A, B] = J_- C^2\}$$

and the corresponding moduli space

$$\mathcal{M}_{J_-}(\Sigma) := M_3 / \operatorname{Stab}(J_-).$$

As in previous sections, let us write

$$C^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

so that

$$[A,B] = J_+C^2 = \begin{pmatrix} -a+c & -b+d \\ -c & -d \end{pmatrix}.$$

We use the traces to stratify M_4 according to the values they take in $\mathbb{C}^2$. Let us write $t_1 = \operatorname{tr} C^2 = a + d$ and $t_2 = \operatorname{tr}[A,B] = -a - d + c$; $c = t_2 + t_1$. We distinguish the special cases when $t_i = \pm 2$, $i = 1, 2$, and also the line of equation $c = 0$, like we did in Section 3.8.

Points $t_1, t_2 = \pm 2$

- $(2, -2)$. In this case $c = 0$, so $a = d = 1$ and $b \in \mathbb{C}$. For $b \neq 0, 1$, C^2 and $[A,B]$ are both of Jordan type and for $b = 0, 1$ either C^2 or $[A,B]$ is equal to Id. We get

$$e(M_3^{1,1}) = (q-2)2e(\overline{X}_3) + 2(\overline{X}_3) + 2(X_1) = 2q^4 + 6q^3 - 6q^2 - 2q.$$

- $(-2, 2)$. Again $c = 0$. C^2 is not of Jordan type if and only if $b = 0$, in which case $C^2 = -\operatorname{Id}$ and $[A,B]$ is of positive Jordan type,

$$e(M_3^{1,2}) = e(C_1 \sqcup C_2)e(\overline{X}_2) = q^5 - q^4 - 5q^3 - 3q^2.$$

- $(2, 2)$. Now $c = 4$ and conjugating by an element in $\operatorname{Stab}(J_-)$ we can assume that $d = 0$, $b = -\frac{1}{c}$ and $a = t_1 = 2$. Both C^2, $[A,B]$ are of positive Jordan type. We obtain

$$e(M_3^{1,3}) = 2qe(\overline{X}_2) = 2q^4 - 4q^3 - 6q^2.$$

- $(-2, 2)$. The stratum is empty since necessarily C^2 is of negative Jordan type.

The total sum is

$$e(M_3^1) = \sum_{i=1}^{3} e(M_3^{1,i}) = q^5 + 3q^4 - 3q^3 - 15q^2 - 2q.$$

Lines $t_i = \pm 2, t_1 = -t_2$

Similar computations to those appearing in the positive Jordan case, Section 3.8, yield the following E-polynomials of M_4 when the traces (t_1, t_2) belong to the lines:

- $\{t_1 = 2, t_2 \neq \pm 2\}$. $e(M_3^{2,1}) = 2qe(\overline{X}_4/\mathbb{Z}_2) = 2q^5 - 4q^4 - 6q^3 + 6q^2 + 2q$.

- $\{t_1 = -2, t_2 \neq \pm 2\}$. The stratum is empty.

- $\{t_2 = 2, t_1 \neq \pm 2\}$. $e(M_3^{2,2}) = qe(\overline{X}_2)e(Z/\mathbb{Z}_2) = q^5 - 5q^4 + 3q^3 + 9q^2$.

- $\{t_2 = -2, t_1 \neq \pm 2\}$ $e(M_3^{2,3}) = qe(\overline{X}_3)e(Z/\mathbb{Z}_2) = q^5 - 9q^3$.

- $\{t_1 = -t_2 \neq \pm 2\}$. This line is characterized by the condition $c = 0$, which implies that $d = a^{-1}$, $b \in \mathbb{C}$. Therefore

$$C^2 = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, \quad [A, B] = \begin{pmatrix} -a & -b + a^{-1} \\ 0 & -a^{-1} \end{pmatrix},$$

where $t_1 = -t_2 = a + a^{-1}$. The fibre is isomorphic to the disjoint union of $\overline{Z}_a \times \overline{X}_{4,-a}$ and $\overline{Z}_{a^{-1}} \times \overline{X}_{4,-a^{-1}}$. Taking the double cover $a \mapsto t_1 = a + a^{-1}$, we get a fibration over $\mathbb{C} \setminus \{0, \pm 1\}$ with fibres $\overline{Z}_a \times \overline{X}_{4,-a}$,which we denote E''. Writing $\sigma'' : \mathbb{C} \mapsto \mathbb{C}$ for the map that takes a to $-a$, we compute its Hodge monodromy representation,

$$\begin{aligned} R(E'') &= R(Z) \otimes R(\sigma''(\overline{X}_4)) = R(Z) \otimes R(\overline{X}_4) \\ &= (T + N) \otimes ((q^3 - 1)T + (3q^2 - 3q)N) \\ &= (q^3 + 3q^2 - 3q - 1)T + (q^3 + 3q^2 - 3q - 1)N. \end{aligned}$$

We obtain

$$\begin{aligned} e(M_3^{2,4}) &= qe(E'') \\ &= q(q - 5)(q^3 + 3q^2 - 3q - 1) \\ &= q^5 - 2q^4 - 18q^3 + 14q^2 + 5q. \end{aligned}$$

Adding all up,

$$e(M_3^2) = \sum_{i=1}^{4} e(M_3^{2,i}) = 5q^5 - 11q^4 - 30q^3 + 29q^2 + 7q.$$

General case

We consider now the case $(t_1, t_2) \in \mathbb{C}^2$, $t_1 \neq -t_2, t_i \neq \pm 2$. If we forget about the condition $t_1 \neq -t_2$, the total space is isomorphic to $\mathbb{C} \times Z/\mathbb{Z}_2 \times \overline{X}_4/\mathbb{Z}_2$; we only need to remove the fibration over the line $(t_1, -t_1)$, with fibre $\mathbb{C} \times Z_a \times \overline{X}_{4,-a}$. This is the space $\overline{M}_1^4/\mathbb{Z}_2$, whose Hodge monodromy representation was computed in Section 3.6. Using Proposition 2.3.6, its E-polynomial is:

$$e(\overline{M}_1^4/\mathbb{Z}_2) = (q - 2)(q^3 + 3q^2) - (q^3 + 3q^2 - 6q - 2) = q^4 - 9q^2 + 6q + 2.$$

By previous considerations

$$e(M_3^3) = q(e(Z/\mathbb{Z}_2)e(\overline{X}_4/\mathbb{Z}_2) - e(\overline{M}_1^4/\mathbb{Z}_2)) = q^6 - 6q^5 + 3q^4 + 21q^3 - 14q^2 - 5q.$$

Reducible orbits

The situation is analogous to the Jordan positive case. The following proposition is proved analogously to Proposition 3.8.1,

Proposition 3.9.1. *$[A, B]$, C^2 share an eigenvector if and only if $c = 0$.*

The reducible locus lies in $\{t_2 = 2\} \cap \{c = 0\}$, i.e, when $(t_1, t_2) = (-2, 2)$, that corresponds to $M_3^{1,2}$. In this case,

$$C^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad [A, B] = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

All such pairs (A, B) are reducible since they are upper-triangular, so the set of reducible orbits will be given by those upper-triangular $C \in SL(2, \mathbb{C})$ such that $C^2 = -\mathrm{Id}$. This set is precisely C_1. The action of U on these orbits is free, as well as in the irreducible ones, so

$$e(M_3^{1,2}/U) = e(C_1 \sqcup C_2)e(\overline{X}_2)/q = q^4 - q^3 - 5q^2 - 3q.$$

The GIT quotient is geometric. Taking the total sum

$$e(M_3) = \sum_{i=1}^{3} e(M_3^i) = q^6 - 5q^4 - 12q^3.$$

and dividing by $e(\mathrm{Stab}\,(J_-)) = q$, we obtain the E-polynomial of the moduli space

$$e(M_{J_-}(\Sigma)) = q^5 - 5q^3 - 12q^2$$

completing Theorem 3.1.2.

Chapter 4

$SL(2,\mathbb{C})$-character varieties of surfaces of genus $g = 3$

4.1 Introduction

This chapter focuses on the computation of the E-polynomials of the character varieties corresponding to a complex curve of genus 3. If we look at the twisted character variety for $g = 3$, $\mathcal{M}^{g=3}_{-\,\mathrm{Id}} = X^{g=3}_1 // SL(2,\mathbb{C})$, which is explicitly the space

$$\begin{aligned}\mathcal{M}^{g=3}_{-\,\mathrm{Id}} = \big\{ &(A_1, B_1, A_2, B_2, A_3, B_3) \in SL(2,\mathbb{C})^6 \mid \\ &[A_1, B_1][A_2, B_2][A_3, B_3] = -\,\mathrm{Id} \big\} // SL(2,\mathbb{C}),\end{aligned}$$

we can mimic the stratification for the representation space done for the genus 2 case. If we write

$$W_i := \sqcup_{i=0}^4 \{(A_1, B_1, A_2, B_2, A_3, B_3) \in SL(2,\mathbb{C})^6 \mid [A_1, B_1][A_2, B_2] = -[B_3, A_3] \in X_i\}$$

we quickly see that $X^{g=2}_1 = \sqcup_{i=0}^4 W_i$. However, as soon as we analyse W_4,

$$\begin{aligned}W_4 := \sqcup_{i=0}^4 \Big\{ &(A_1, B_1, A_2, B_2, A_3, B_3) \in SL(2,\mathbb{C})^6 \mid \\ &[A_1, B_1][A_2, B_2] = -[B_3, A_3] \sim \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \lambda \neq 0, \pm 1 \Big\}\end{aligned}$$

we observe that it is required to understand the behaviour of the parabolic character variety of genus 2, when λ varies in $\mathbb{C}^* \setminus \{\pm 1\}$. In other words, to deal with the genus 3 case using the fibration techniques appearing in Chapter 2, the Hodge monodromies of the following fibrations are needed:

$$\overline{X}^{g=2}_4 \longrightarrow \mathbb{C}^* \setminus \{\pm 1\} \tag{4.1}$$

$$\overline{X}^{g=2}_4 / \mathbb{Z}_2 \longrightarrow \mathbb{C} \setminus \{\pm 2\} \tag{4.2}$$

We explicitly compute $R(\overline{X}_4^{g=2})$ in Section 4.3, analysing the behaviour of the stratification of $\overline{X}_4^{g=2}$ given in [53], showing that it is compatible with the monodromy. However, the same idea for $R(\overline{X}_4/\mathbb{Z}_2)$ gets much more difficult due to both the complexity of the strata and the $\mathbb{Z}_2$-action. We compute $R(\overline{X}_4/\mathbb{Z}_2)$ instead using an indirect approach, which inspires how to tackle the general genus case.

The idea is the following: if we write $R(\overline{X}_4^{g=2}) = aT + bS_2 + cS_{-2} + dS_0$, we show in Section 4.3 that we can deduce the missing data $a, b, c, d \in \mathbb{Z}[q]$ from $R(\overline{X}_4), e(\overline{X}_4^{g=2})$ and $e(\overline{X}_1^{g=3})$. In order to compute the latter without $R(\overline{X}_4/\mathbb{Z}_2)$, we use the trace map of the commutators $\mathcal{M}(\mathrm{SL}(2,\mathbb{C})) \to \mathbb{C}^3$ given by $(A_1, B_1, A_2, B_2, A_3, B_3) \mapsto (t_1, t_2, t_3)$, $t_i = \mathrm{tr}([A_i, B_i])$ to stratify the representation space (see Section 4.2). This is the most technical part of the chapter and here is where we use Hodge monodromy representations over a base of dimension bigger that 1. Finally, $R(\overline{X}_4/\mathbb{Z}_2)$ allows us to compute $e(X_0^{g=3})$ and $e(\mathcal{M}_{\mathrm{Id}}^{g=3})$, after identifying the correct orbits in the reducible locus in order to obtain the GIT quotient. The computations in this chapter are used as the starting point for the induction for general genus developed in Chapter 4.

The main results of the chapter are the following:

Theorem 4.1.1. *Let X be a complex curve of genus $g = 3$. Then the E-polynomials of the character varieties are:*

$$e(\mathcal{M}_{\mathrm{Id}}^{g=3}) = q^{12} - 4q^{10} + 74q^8 + 375q^6 + 16q^4 + q^2 + 1\,,$$
$$e(\mathcal{M}_{\mathrm{Id}}^{g=3}) = q^{12} - 4q^{10} + 6q^8 - 252q^7 - 14q^6 - 252q^5 + 6q^4 - 4q^2 + 1$$

where $q = uv$.

Proposition 4.1.2. *The Hodge monodromy representation of $R(\overline{X}_4^{g=2}/\mathbb{Z}_2)$ is:*

$$R(\overline{X}_4^{g=2}/\mathbb{Z}_2) = (q^9 - 3q^7 + 6q^5)T - (45q^5 + 15q^3)S_2 + (15q^6 + 45q^4)S_{-2} + (-6q^4 + 3q^2 - 1)S_0$$

As a byproduct, we obtain the behaviour of the E-polynomial of the parabolic character variety ($G = \mathrm{SL}(2,\mathbb{C})$)

$$\mathcal{M}^\lambda(G) = \left\{ (A_1, B_1, \ldots, A_g, B_g) \in G^{2g} \,|\, \prod_{i=1}^{g} [A_i, B_i] = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \right\} //G.$$

when λ varies in $\mathbb{C} - \{0, \pm 1\}$ for the case $g = 2$. This is given by the following formula

$$R(\mathcal{M}^\lambda) = (q^8 + q^7 - 2q^6 - 2q^5 + 4q^4 - 2q^3 - 2q^2 + q + 1)T + 15(q^5 - 2q^4 + q^3)N. \quad (4.3)$$

which means that the E-polynomial of the invariant part of the cohomology is the polynomial accompanying T, and the E-polynomial of the non-invariant part is the polynomial accompanying N.

4.2 E-polynomial of the twisted character variety

We start by computing the E-polynomial of the twisted character variety $\mathcal{M}_{-\mathrm{Id}}$ for a curve X of genus 3. This space can be described as the quotient

$$\mathcal{M}^1 = W/\operatorname{PGL}(2,\mathbb{C}), \tag{4.4}$$

where

$$W = \{(A_1,B_1,A_2,B_2,A_3,B_3) \in \mathrm{SL}(2,\mathbb{C})^6 \mid [A_1,B_1][A_2,B_2][A_3,B_3] = -\mathrm{Id}\}.$$

Notice that we only need to consider a geometric quotient, since all representations are irreducible: if there was a common eigenvector v for $(A_1,B_1,A_2,B_2,A_3,B_3)$, we would obtain that $[A_1,B_1](v) = [A_2,B_2](v) = [A_3,B_3](v) = v$ and therefore $[A_1,B_1][A_2,B_2][A_3,B_3](v) = v \neq -\mathrm{Id}(v) = -v$.

As we mentioned in the introduction, we will stratify the space into locally closed subvarieties according to the possible values of the traces of the commutators. To simplify the notation, we write $[A_1,B_1] = \xi_1, [A_2,B_2] = \xi_2, [A_3,B_3] = \xi_3$. Consider the map

$$\begin{aligned} F : W &\longrightarrow \mathbb{C}^3 \\ (A_1,B_1,A_2,B_2,A_3,B_3) &\mapsto (t_1,t_2,t_3) = (\operatorname{tr}\xi_1, \operatorname{tr}\xi_2, \operatorname{tr}\xi_3). \end{aligned}$$

We are interested in the following condition: if ξ_1,ξ_2,ξ_3 share an eigenvector v with eigenvalue λ_i in each case, then in a suitable basis

$$[A_1,B_1] = \begin{pmatrix} \lambda_1 & * \\ 0 & \lambda_1^{-1} \end{pmatrix}, \quad [A_2,B_2] = \begin{pmatrix} \lambda_2 & * \\ 0 & \lambda_2^{-1} \end{pmatrix}, \quad [A_3,B_3] = \begin{pmatrix} \lambda_3 & * \\ 0 & \lambda_3^{-1} \end{pmatrix},$$

and therefore, as $\xi_1\xi_2\xi_3 = -\mathrm{Id}$, we obtain that $\lambda_1\lambda_2\lambda_3 = -1$. Other possibility is that v has eigenvalue λ_1 for ξ_1, λ_2 for ξ_2 and λ_3^{-1} for ξ_3, yielding $\lambda_1\lambda_2\lambda_3^{-1} = -1$, etc. Working out all possibilities, we have that ξ_1,ξ_2,ξ_3 can share an eigenvector when

$$\lambda_3 = -\lambda_1\lambda_2,\ \lambda_3 = -\lambda_1^{-1}\lambda_2,\ \lambda_3 = -\lambda_1\lambda_2^{-1},\ \text{or } \lambda_3 = -\lambda_1^{-1}\lambda_2^{-1} \tag{4.5}$$

Equivalently, in terms of the traces, when

$$t_1^2 + t_2^2 + t_3^2 + t_1t_2t_3 = 4. \tag{4.6}$$

This defines a (smooth) cubic surface $C \subset \mathbb{C}^3$, depicted in Figure 4.1.

Lemma 4.2.1. *ξ_1,ξ_2,ξ_3 share an eigenvector if and only if $(t_1,t_2,t_3) \in C$.*

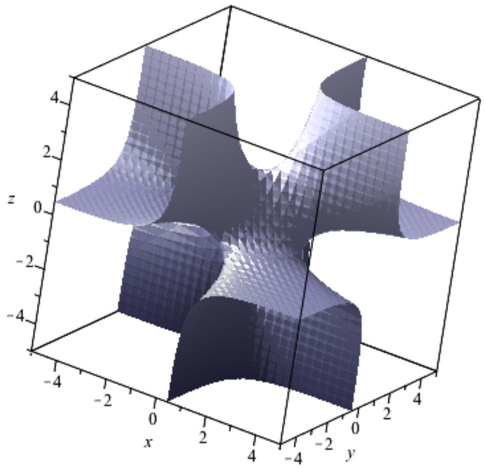

Figure 4.1: The surface C.

Proof. We have already seen the if part. For the only if part, note that if (t_1, t_2, t_3) satisfies (4.6) then the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ satisfy one of the four relations in (4.5). Changing some λ_i by λ_i^{-1} if necessary, we can suppose that $\lambda_3 = -\lambda_1\lambda_2$.

Suppose that some ξ_i is diagonalizable. Without loss of generality, we suppose it is ξ_1, and choose a basis with $\xi_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix}$. Write $\xi_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then we have the equation

$$\xi_3^{-1} = -\xi_1\xi_2 = -\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = -\begin{pmatrix} \lambda_1 a & \lambda_1 b \\ \lambda_1^{-1} c & \lambda_1^{-1} d \end{pmatrix}.$$

We get the equations $t_2 = a + d$ and $t_3 = \operatorname{tr} \xi_3 = \operatorname{tr} \xi_3^{-1} = -\lambda_1 a - \lambda_1^{-1} d$. Hence $\lambda_2 + \lambda_2^{-1} = a + d$ and $\lambda_1\lambda_2 + \lambda_1^{-1}\lambda_2^{-1} = \lambda_1 a + \lambda_1^{-1} d$. If $\lambda_1 \neq \pm 1$, it must be $a = \lambda_2, d = \lambda_2^{-1}$; thus $ad = 1$ and hence $bc = 0$, which implies that the matrices share an eigenvector. If $\lambda_1 = \pm 1$, then $\xi_1 = \pm \operatorname{Id}$, and $\xi_2 = \pm \xi_3$, so the matrices share their eigenvectors.

Now suppose that none of the ξ_i are diagonalizable, so they are of Jordan type. Let v_i be the only eigenvector (up to scalar multiples) of ξ_i. If ξ_i do not share an eigenvector, then v_1, v_2 is a basis, on which $\xi_1 = \lambda_1 \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, $\xi_2 = \lambda_2 \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$, where $\lambda_i \in \{\pm 1\}$. Then $\xi_3^{-1} = -\lambda_1\lambda_2 \begin{pmatrix} 1 + bc & b \\ c & 1 \end{pmatrix}$ hence $t_3 = 2\lambda_3 = -2\lambda_1\lambda_2 = -\lambda_1\lambda_2(2 + bc)$. So $bc = 0$, which is a contradiction. $\square$

We stratify the space W according to the traces of ξ_1, ξ_2, ξ_3. Consider the planes $t_i = \pm 2$ and the cubic C above. Then consider as well the intersections of these seven subvarieties. This gives the required stratification. We shall compute the E-polynomials of the chunk of W lying above each of these strata, starting by the lower-dimensional ones (points) and going up in dimension.

We shall use some of the polynomials in [53] that correspond to some of the strata of the spaces of representations for the case of a curve of genus $g = 2$. We shall point out which stratum we use each time.

4.2.1 Special points

The intersections of three planes, or of the cubic surface and two planes, is the collection of eight points given by $(t_1, t_2, t_3) = (\pm 2, \pm 2, \pm 2)$. Let W_1 be the subset of $(A_1, B_1, A_2, B_2, A_3, B_3) \in W$ with traces given by these possibilities. Note that if $t_i = 2$ then $\xi_i = \mathrm{Id}$ or ξ_i is of Jordan type J_+; analogously, if $t_i = -2$ then $\xi_i = -\mathrm{Id}$ or ξ_i is of Jordan type J_-.

- $W_{11} = \{(t_1, t_2, t_3) = (2, 2, 2)\} = F^{-1}((2, 2, 2))$. Then ξ_i are all of Jordan type (if $\xi_1 = \mathrm{Id}$ then $\xi_2\xi_3 = -\mathrm{Id}$, so $\xi_2 = -\xi_3^{-1}$ and $t_2 = -t_3$). Choosing an adequate basis, we can assume that $[A_1, B_1] = J_+$, so $[A_2, B_2][A_3, B_3] = -(J_+)^{-1} = J_-$. This set has been determined in [53, Stratum Z_3, Section 12]. It has polynomial $q\, e(\overline{X}_2)^2$. Therefore

$$\begin{aligned} e(W_{11}) &= q\, e(\overline{X}_2)^3 e(\mathrm{PGL}(2, \mathbb{C})/U) \\ &= q^{12} - 6q^{11} + 2q^{10} + 34q^9 - 12q^8 - 82q^7 - 18q^6 + 54q^5 + 27q^4. \end{aligned}$$

- $W_{12} = \{(t_1, t_2, t_3) = (2, 2, -2)\} = F^{-1}((2, 2, -2))$ and the cyclic permutations (accounting for three cases). If $\xi_1 = \mathrm{Id}$, then $[A_2, B_2][A_3, B_3] = -\mathrm{Id}$, with $t_2 = 2$, and $t_3 = -2$. If $\xi_2 = \mathrm{Id}$ then $\xi_3 = -\mathrm{Id}$; and if $\xi_2 \sim J_+$ then $\xi_3 \sim J_-$. This produces the contribution (multiplying by 3 because of the three cyclic permutations)

$$\begin{aligned} e(W'_{12}) &= 3e(X_0)(e(X_0)e(X_1) + e(\mathrm{PGL}(2, \mathbb{C})/U)e(\overline{X}_2)e(\overline{X}_3)) \\ &= 3q^{12} + 18q^{11} + 3q^{10} - 126q^9 - 147q^8 \\ &\quad + 150q^7 + 273q^6 + 6q^5 - 132q^4 - 48q^3. \end{aligned}$$

  If $\xi_1 \sim J_+$, then choosing an adequate basis, we can assume that $[A_1, B_1] = J_+$, so $[A_2, B_2][A_3, B_3] = -(J_+)^{-1} = J_-$. This set has been determined in [53, Stratum Z_1, Section 12] to be $(q-2)e(\overline{X}_2)e(\overline{X}_3) + e(\overline{X}_2)e(X_1) + e(\overline{X}_3)e(X_0)$. Therefore

$$\begin{aligned} e(W''_{12}) &= 3e(\overline{X}_2)e(\mathrm{PGL}(2, \mathbb{C})/U)\big((q-2)e(\overline{X}_2)e(\overline{X}_3) \\ &\quad + e(\overline{X}_2)e(X_1) + e(\overline{X}_3)e(X_0)\big) \\ &= 6q^{12} + 9q^{11} - 72q^{10} - 66q^9 + 120q^8 \\ &\quad + 36q^7 - 144q^6 - 6q^5 + 90q^4 + 27q^3. \end{aligned}$$

Finally

$$\begin{aligned} e(W_{12}) &= e(W'_{12}) + e(W''_{12}) \\ &= 9q^{12} + 27q^{11} - 69q^{10} - 192q^9 - 27q^8 + 186q^7 + 129q^6 - 42q^4 - 21q^3. \end{aligned}$$

- $W_{13} = \{(t_1, t_2, t_3) = (-2, -2, 2)\} = F^{-1}((-2, -2, 2))$ and cyclic permutations (which account for three cases). If $\xi_1 = -\mathrm{Id}$ then it must be $t_2 = t_3$, which is not the case. Therefore ξ_1 is of Jordan type J_-. Choosing a suitable basis, we can write $\xi_1 = J_-$, so $[A_1, B_1][A_2, B_2] = -J_-^{-1} = J_+$. This set has been determined in [53, Stratum Z_3, Section 11] and it has polynomial $q\, e(\overline{X}_2)e(\overline{X}_3)$. Therefore

$$\begin{aligned} e(W_{13}) &= 3q\, e(\overline{X}_3)^2 e(\overline{X}_2) e(\mathrm{PGL}(2, \mathbb{C})/U) \\ &= 3q^{12} + 12q^{11} - 21q^{10} - 120q^9 - 63q^8 + 108q^7 + 81q^6. \end{aligned}$$

- $W_{14} = \{(t_1, t_2, t_3) = (-2, -2, -2)\} = F^{-1}((-2, -2, -2))$. If $\xi_1 = -\mathrm{Id}$, then $[A_2, B_2][A_3, B_3] = \mathrm{Id}$, with $t_2 = -2$, and $t_3 = -2$. If $\xi_2 = -\mathrm{Id}$ then $\xi_3 = -\mathrm{Id}$; and if $\xi_2 \sim J_-$ then $\xi_3 \sim J_-$. This produces the contribution

$$\begin{aligned} e(W'_{14}) &= e(X_1)\left(e(X_1)^2 + e(\mathrm{PGL}(2, \mathbb{C})/U) e(\overline{X}_3)^2\right) \\ &= q^{11} + 6q^{10} + 8q^9 - 12q^8 - 20q^7 + 6q^6 + 12q^5 - q^3. \end{aligned}$$

If $\xi_1 \sim J_-$, then choosing an adequate basis, we can assume that $[A_1, B_1] = J_-$, so $[A_2, B_2][A_3, B_3] = -(J_-)^{-1} = J_+$. This set has been determined in [53, Stratum Z_2, Section 11] to be $(q-2)e(\overline{X}_3)^2 + 2e(\overline{X}_3)e(X_1)$. Therefore

$$\begin{aligned} e(W''_{14}) &= e(\overline{X}_3)\left((q-2)e(\overline{X}_3)^2 + 2e(\overline{X}_3)e(X_1)\right) e(\mathrm{PGL}(2, \mathbb{C})/U) \\ &= q^{12} + 9q^{11} + 20q^{10} - 20q^9 - 87q^8 - 7q^7 + 66q^6 + 18q^5. \end{aligned}$$

Hence

$$\begin{aligned} e(W_{14}) &= e(W'_{14}) + e(W''_{14}) \\ &= q^{12} + 10q^{11} + 26q^{10} - 12q^9 - 99q^8 - 27q^7 + 72q^6 + 30q^5 - q^3. \end{aligned}$$

Adding all up, we obtain

$$e(W_1) = 14q^{12} + 43q^{11} - 62q^{10} - 290q^9 - 201q^8 + 185q^7 + 264q^6 + 84q^5 - 15q^4 - 22q^3.$$

4.2.2 Special lines

The intersection of two of the planes are the lines $\{(t_1, t_2, t_3) = (\pm 2, \pm 2, t_3), t_3 \in \mathbb{C} - \{\pm 2\}\}$, and the cyclic permutations of these. The intersection of one of the planes and the surface is given by the lines $\{t_1 = \pm 2,\ t_2 = \mp t_3 \in \mathbb{C} - \{\pm 2\}\}$ and the cyclic permutations. We denote by W_2 the portion of W lying over these lines, and we stratify in the following sets:

- W_{21} given by $t_1 = 2$, $t_2 = \pm 2$ and $t_3 \in \mathbb{C} - \{\pm 2\}$. Note that if $\xi_1 = \mathrm{Id}$ then $\xi_2\xi_3 = -\mathrm{Id}$, with $\mathrm{tr}\,\xi_2 = \pm 2$ and $\mathrm{tr}\,\xi_3 \neq \pm 2$, a contradiction. Then ξ_1 is of Jordan type. Choosing an adequate basis, we can assume that $\xi_1 = J_+$. Then $[A_2, B_2][A_3, B_3] = -(J_+)^{-1} = J_-$, where $\mathrm{tr}\,\xi_2 = \pm 2$ and $\mathrm{tr}\,\xi_3 \neq \pm 2$. This set has been computed in [53, Stratum Z_4, Section 12] and it has E-polynomial $q\,(e(\overline{X}_2) + e(\overline{X}_3))e(\overline{X}_4/\mathbb{Z}_2)$. So

$$\begin{aligned} e(W_{21}) &= e(\overline{X}_2)e(\mathrm{PGL}(2,\mathbb{C})/U)q\,(e(\overline{X}_2) + e(\overline{X}_3))e(\overline{X}_4/\mathbb{Z}_2) \\ &= 2q^{13} - 7q^{12} - 13q^{11} + 47q^{10} + 40q^9 \\ &\quad - 103q^8 - 58q^7 + 93q^6 + 38q^5 - 30q^4 - 9q^3. \end{aligned}$$

- W_{22} given by $t_1 = -2$, $t_2 = \pm 2$ and $t_3 \in \mathbb{C} - \{\pm 2\}$. Again ξ_1 is of Jordan type. Fixing a basis we have $\xi_1 = J_-$ and $\xi_2\xi_3 = -J_-^{-1} = J_+$, where $\mathrm{tr}\,\xi_2 = \pm 2$ and $\mathrm{tr}\,\xi_3 \neq \pm 2$. This set has been computed in [53, Stratum Z_4, Section 11] and it has E-polynomial $q\,(e(\overline{X}_2) + e(\overline{X}_3))e(\overline{X}_4/\mathbb{Z}_2)$. So

$$\begin{aligned} e(W_{22}) &= e(\overline{X}_3)e(\mathrm{PGL}(2,\mathbb{C})/U)q(e(\overline{X}_2) + e(\overline{X}_3))e(\overline{X}_4/\mathbb{Z}_2) \\ &= 2q^{13} + 3q^{12} - 22q^{11} - 27q^{10} + 61q^9 + 58q^8 - 68q^7 - 43q^6 + 27q^5 + 9q^4. \end{aligned}$$

- W_{23} given by $t_1 = 2$, $\xi_1 = \mathrm{Id}$ and $t_2 = -t_3 \in \mathbb{C} - \{\pm 2\}$. Now $\xi_2\xi_3 = -\mathrm{Id}$, $t_2 = -t_3 \neq \pm 2$. This is computed in [53, Stratum W_4, Section 9], $e(W_4) = q^9 - 2q^8 - 7q^7 - 18q^6 + 24q^5 + 28q^4 - 17q^3 - 8q^2 - q$. So

$$\begin{aligned} e(W_{23}) &= e(X_0)e(W_4) \\ &= q^{13} + 2q^{12} - 16q^{11} - 48q^{10} - 33q^9 + 170q^8 \\ &\quad + 143q^7 - 200q^6 - 128q^5 + 72q^4 + 33q^3 + 4q^2. \end{aligned}$$

- W_{24} given by $t_1 = 2$, $\xi_1 \sim J_+$. Using an adequate basis, we have $\xi_1 = J_+$, and $\xi_2\xi_3 = -J_+^{-1} = J_-$, and $t_2 = -t_3 \neq \pm 2$. This is computed in [53, Stratum Z_5, Section 12], $e(Z_5) = q^8 - 3q^7 - 3q^6 - 35q^5 + 69q^4 - 15q^3 - 11q^2 - 3q$. So

$$\begin{aligned} e(W_{24}) &= e(\overline{X}_2)e(\mathrm{PGL}(2,\mathbb{C})/U)e(Z_5) \\ &= q^{13} - 5q^{12} - q^{11} - 15q^{10} + 148q^9 \\ &\quad - 28q^8 - 336q^7 + 112q^6 + 227q^5 - 55q^4 - 39q^3 - 9q^2. \end{aligned}$$

- W_{25} given by $t_1 = -2$, $\xi_1 = -\mathrm{Id}$. Then $\xi_2\xi_3 = \mathrm{Id}$, with $t_2 = t_3 \neq \pm 2$. This is computed in [53, Stratum Y_4, Section 8],[1] $e(Y_4) = q^9 - 2q^8 + 2q^7 - 18q^6 + 6q^5 + 28q^4 - 8q^3 - 8q^2 - q$. So

$$\begin{aligned} e(W_{25}) &= e(X_1)e(Y_4) \\ &= q^{12} - 2q^{11} + q^{10} - 16q^9 + 4q^8 + 46q^7 - 14q^6 - 36q^5 + 7q^4 + 8q^3 + q^2. \end{aligned}$$

- W_{26} given by $t_1 = -2$, $\xi_1 \sim J_-$. Choosing a basis so that $\xi_1 = J_-$, $\xi_2\xi_3 = J_+$, with $t_2 = t_3 \neq \pm 2$. This is computed in [53, Stratum Z_5, Section 11], $e(Z_5) = q^8 - 3q^7 - 3q^6 - 35q^5 + 69q^4 - 15q^3 - 11q^2 - 3q$. So

$$\begin{aligned} e(W_{26}) &= e(\overline{X}_3)e(\mathrm{PGL}(2,\mathbb{C})/U)e(Z_5) \\ &= q^{13} - 13q^{11} - 44q^{10} - 24q^9 \\ &\quad + 236q^8 - 20q^7 - 228q^6 + 47q^5 + 36q^4 + 9q^3. \end{aligned}$$

Considering the possible permutations, and adding the E-polynomials of the strata, we get

$$\begin{aligned} e(W_2) &= 3e(W_{21}) + 3e(W_{22}) + 3e(W_{23}) + 3e(W_{24}) + 3e(W_{25}) + 3e(W_{26}) \\ &= 21q^{13} - 18q^{12} - 201q^{11} - 258q^{10} + 528q^9 \\ &\quad + 1011q^8 - 879q^7 - 840q^6 + 525q^5 + 117q^4 + 6q^3 - 12q^2. \end{aligned}$$

4.2.3 Special planes

Now consider the planes given by the equations $\{t_i = \pm 2\}$, from which we remove the previous strata. We do the case $t_1 = \pm 2$ and multiply by three the result to account for the three cases $i = 1, 2, 3$. The planes are given by $t_1 = \pm 2$, where $t_2, t_3 \neq \pm 2$ and $t_2 \neq \mp t_3$. Note that it cannot be $\xi_1 = \pm\mathrm{Id}$, since this would imply $t_2 = \mp t_3$. We have the following cases:

- W_{31} given by $\xi_1 \sim J_+$. This implies that, in a suitable basis, $\xi_2\xi_3 = J_-$, together with the previous restrictions for the traces. This is the set given in [53, Stratum Z_6, Section 12], which has E-polynomial $e(Z_6) = q^9 - 5q^8 + 24q^6 + 20q^5 - 60q^4 + 6q^3 + 11q^2 + 3q$. So

$$\begin{aligned} e(W_{31}) &= e(\overline{X}_2)e(\mathrm{PGL}(2,\mathbb{C})/U)e(Z_6) \\ &= q^{14} - 7q^{13} + 6q^{12} + 46q^{11} - 35q^{10} \\ &\quad - 211q^9 + 94q^8 + 351q^7 - 103q^6 - 218q^5 + 28q^4 + 39q^3 + 9q^2. \end{aligned}$$

[1] We make a correction to $e(Y_4)$ in [53], $e(Y_4) = q^9 - 2q^8 + 2q^7 - 18q^6 + 6q^5 + 28q^4 - 8q^3 - 8q^2 - q$.

- W_{32} given by $\xi_1 \sim J_-$. This implies that, in a suitable basis, $\xi_2\xi_3 = J_+$, together with the restrictions $t_2, t_3 \neq \pm 2$, $t_2 \neq t_3$. This is the set given in [53, Stratum Z_6, Section 11], which has E-polynomial $e(Z_6) = q^9 - 5q^8 + 15q^6 + 11q^5 - 51q^4 + 15q^3 + 11q^2 + 3q$. So

$$\begin{aligned} e(W_{32}) &= e(\overline{X}_3)e(\mathrm{PGL}(2,\mathbb{C})/U)e(Z_6) \\ &= q^{14} - 2q^{13} - 16q^{12} + 17q^{11} + 71q^{10} \\ &\quad - 33q^9 - 194q^8 + 74q^7 + 174q^6 - 47q^5 - 36q^4 - 9q^3. \end{aligned}$$

The total contribution of the planes is

$$\begin{aligned} e(W_3) &= 3(e(W_{31}) + e(W_{32})) \\ &= 6q^{14} - 27q^{13} - 30q^{12} + 189q^{11} + 108q^{10} - 732q^9 \\ &\quad - 300q^8 + 1275q^7 + 213q^6 - 795q^5 - 24q^4 + 90q^3 + 27q^2. \end{aligned}$$

4.2.4 The cubic surface C. One eigenvector

We now deal with the part of W lying over the cubic surface C, with $t_1, t_2, t_3 \neq \pm 2$. As we saw in Lemma 4.2.1, this corresponds to the case that ξ_1, ξ_2, ξ_3 share (at least) one eigenvector.

We deal now with the case where ξ_1, ξ_2 and ξ_3 share just one eigenvector (up to scalar multiples), which we denote by v. Using it as the first vector of a basis that diagonalizes ξ_1, we can arrange that

$$\xi_1 = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix},\ \xi_2 = \begin{pmatrix} \mu & 1 \\ 0 & \mu^{-1} \end{pmatrix},\ \xi_3 = \begin{pmatrix} -\lambda^{-1}\mu^{-1} & \lambda \\ 0 & -\lambda\mu \end{pmatrix} \tag{4.7}$$

where $\lambda = \lambda_1$, $\mu = \lambda_2$ are the eigenvalues of ξ_1, ξ_2 associated to the eigenvector v. Our choice of basis for the above expressions of ξ_1, ξ_2, ξ_3 gives us a slice for the $\mathrm{PGL}(2,\mathbb{C})$-action, since their stabilizer is trivial. So we shall only need to multiply by $e(\mathrm{PGL}(2,\mathbb{C}))$ after computing the E-polynomial of the space

$$\{(A_1, B_1, A_2, B_2, A_3, B_3) \mid [A_i, B_i] = \xi_i, i = 1 \ldots 3\}$$

where ξ_1, ξ_2, ξ_3 satisfy (4.7). This set can be regarded as a fibration

$$Z \longrightarrow B = \{(\lambda, \mu) \in (\mathbb{C}^*)^2 \mid \lambda, \mu, \lambda\mu \neq \pm 1\},$$

with fiber

$$\overline{X}_{4,\lambda} \times \overline{X}_{4,\mu} \times \overline{X}_{4,-\lambda^{-1}\mu^{-1}}.$$

To compute its E-polynomial, we would like to extend the fibration to the six curves $\lambda, \mu, \lambda\mu = \pm 1$ and apply Corollary 2.3.7. We cannot extend the fibration; however we can extend the local system defining the Hodge monodromy fibration. By (3.2), the Hodge monodromy fibration $R(\overline{X}_4)$ is trivial over $\lambda = \pm 1$, and it is of order 2 over $\lambda = 0$. Consider the projections:

$$\begin{array}{rcl} \pi_1 : B & \longrightarrow & \mathbb{C}^* - \{\pm 1\} \\ (\lambda, \mu) & \mapsto & \lambda \\ \pi_2 : B & \longrightarrow & \mathbb{C}^* - \{\pm 1\} \\ (\lambda, \mu) & \mapsto & \mu \\ \pi_3 : B & \longrightarrow & \mathbb{C}^* - \{\pm 1\} \\ (\lambda, \mu) & \mapsto & -\lambda^{-1}\mu^{-1} \end{array}$$

Then

$$\overline{Z} = \pi_1^*(\overline{X}_4) \times \pi_2^*(\overline{X}_4) \times \pi_3^*(\overline{X}_4).$$

The Hodge monodromy fibration $R(\overline{Z})$ can be extended (locally trivially) over the lines $\lambda = \pm 1$, $\mu = \pm 1$ and $\lambda\mu = \pm 1$, to a Hodge monodromy fibration $\tilde{R}(\overline{Z})$ over $\tilde{B} = \mathbb{C}^* \times \mathbb{C}^*$. Moreover, the monodromy around $\lambda = 0$ and $\mu = 0$ is of order two. The corresponding group is $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$ and the representation ring is generated by representations $T, N_1, N_2, N_{12} = N_1 \otimes N_2$, where T is the trivial representation, N_1 is the representation with non-trivial monodromy around the origin of the first copy of $\mathbb{C}^*$, and N_2 is the representation with non-trivial monodromy around the origin of the second copy of $\mathbb{C}^*$.

Pulling back the Hodge monodromy representation of $\overline{X}_4$ given in (3.2), we have that $R(\pi_1^*(\overline{X}_4)) = aT + bN_1$, $R(\pi_2^*(\overline{X}_4)) = aT + bN_2$ and $R(\pi_3^*(\overline{X}_4)) = aT + bN_{12}$, where $a = q^3 - 1$, $b = 3q^2 - 3q$. So the Hodge monodromy representation of $\overline{Z}$ is

$$\begin{aligned} R(\overline{Z}) &= (aT + bN_1) \otimes (aT + bN_2) \otimes (aT + bN_{12}) \\ &= (a^3 + b^3)T + (a^2 b + ab^2)N_1 + (a^2 b + ab^2)N_2 + (a^2 b + ab^2)N_{12}. \end{aligned}$$

We extend $R(\overline{Z})$ to a Hodge monodromy fibration $\tilde{R}(\overline{Z})$ over $\tilde{B} = \mathbb{C}^* \times \mathbb{C}^*$ with the same formula, and compute its E-polynomial, applying Corollary 2.3.7,

$$\begin{aligned} e(\tilde{R}(\overline{Z})) &= (q-1)^2 e(F)^{inv} = (q-1)^2(a^3 + b^3) = (q-1)^2((q^3-1)^3 + (3q^2 - 3q)^3) \\ &= q^{11} - 2q^{10} + q^9 + 24q^8 - 129q^7 \\ &+ 267q^6 - 267q^5 + 129q^4 - 24q^3 - q^2 + 2q - 1. \end{aligned}$$

Now we substract the contribution of $\tilde{R}(\overline{Z})$ over the lines $\lambda = \pm 1$, $\mu = \pm 1$ and $\lambda\mu = \pm 1$.

- Consider the curve defined by $\lambda = 1$, $\mu \neq \pm 1$. The fibration has the Hodge monodromy of a fibration over $\mathbb{C}^* - \{\pm 1\}$ with fiber

$$\overline{X}_{4,\lambda_0} \times \overline{X}_{4,\mu} \times \overline{X}_{4,-\mu^{-1}}$$

This has Hodge monodromy representation equal to $e(\overline{X}_{4,\lambda_0})R(\overline{X}_4) \otimes \tau^* R(\overline{X}_4)$, where $\tau(\mu) = -\mu^{-1}$. This is equal to

$$\begin{aligned} e(\overline{X}_{4,\lambda_0})R(\overline{X}_4) \otimes \tau^* R(\overline{X}_4) &= e(\overline{X}_{4,\lambda_0})R(\overline{X}_4) \otimes R(\overline{X}_4) \\ &= e(\overline{X}_{4,\lambda_0})(aT + bN) \otimes (aT + bN) \\ &= e(\overline{X}_{4,\lambda_0})((a^2 + b^2)T + (2ab)N). \end{aligned}$$

Using Corollary 2.3.5, we get that the contribution equals

$$\begin{aligned} &e(\overline{X}_{4,\lambda_0}) \big((q-3)(a^2+b^2) - 2(2ab)\big) \\ &\quad = q^{10} - 15q^8 - 36q^7 - 24q^6 + 300q^5 - 238q^4 - 60q^3 + 39q^2 + 20q + 3. \end{aligned} \tag{4.8}$$

- The computation for $\lambda = -1$, $\mu \neq \pm 1$ is analogous and gives the same quantity.

- By symmetry, the contribution for $\mu = \pm 1$, and for $\lambda\mu = \pm 1$ is the same as for $\lambda = \pm 1$. So we have to multiply (4.8) by 6.

- The contribution of the four points $(\pm 1, \pm 1)$ is $4e(X_{4,\lambda_0})^3 = 4q^9 + 36q^8 + 72q^7 - 120q^6 - 288q^5 + 288q^4 + 120q^3 - 72q^2 - 36q - 4$.

Therefore, the E-polynomial of the original fibration is:

$$\begin{aligned} e(\overline{Z}) = e(\tilde{R}(\overline{Z})) &- 6(q^{10} - 15q^8 - 36q^7 - 24q^6 + 300q^5 \\ &- 238q^4 - 60q^3 + 39q^2 + 20q + 3) - 4e(X_{4,\lambda_0})^3 \\ = q^{11} &- 8q^{10} - 3q^9 + 78q^8 + 15q^7 + 531q^6 \\ &- 1779q^5 + 1209q^4 + 216q^3 - 163q^2 - 82q - 15 \end{aligned}$$

and

$$\begin{aligned} e(W_4) &= e(\mathrm{PGL}(2,\mathbb{C}))e(\overline{Z}) \\ &= q^{14} - 8q^{13} - 4q^{12} + 86q^{11} + 18q^{10} + 453q^9 - 1794q^8 \\ &\quad + 678q^7 + 1995q^6 - 1372q^5 - 298q^4 + 148q^3 + 82q^2 + 15q. \end{aligned}$$

4.2.5 The cubic surface C. Two eigenvectors

Suppose now that $(t_1, t_2, t_3) \in C$ and ξ_1, ξ_2, ξ_3 share two eigenvectors. Then they can all be simultaneously diagonalized. With respect to a suitable basis, we have that:

$$[A_1, B_1] = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad [A_2, B_2] = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}, \quad [A_3, B_3] = \begin{pmatrix} -\lambda^{-1}\mu^{-1} & 0 \\ 0 & -\lambda\mu \end{pmatrix}.$$

This defines a fibration $\overline{Z} \to B := \{(\lambda, \mu) \in (\mathbb{C}^*)^2 | \lambda, \mu, \lambda\mu \neq \pm 1\}$, with fiber

$$\overline{X}_{4,\lambda} \times \overline{X}_{4,\mu} \times \overline{X}_{4,-\lambda^{-1}\mu^{-1}}$$

The stabilizer of ξ_1, ξ_2, ξ_3 is $D \times \mathbb{Z}_2$, where D are the diagonal matrices and the $\mathbb{Z}_2$-action is given by the simultaneous permutation of the eigenvalues, i.e, by conjugation by $P_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Therefore the stratum we are analysing is

$$W_5 \cong Z = (\overline{Z} \times \mathrm{PGL}(2, \mathbb{C})/D)/\mathbb{Z}_2\,.$$

The action on the basis of $\overline{Z} \to B$ takes (λ, μ) to (λ^{-1}, μ^{-1}), producing a fibration

$$\begin{array}{ccc} \overline{Z} & \longrightarrow & \overline{Z}' := \overline{Z}/\mathbb{Z}_2 \\ \downarrow & & \downarrow \\ B & \longrightarrow & B' = B/\mathbb{Z}_2 \end{array} \tag{4.9}$$

If we write

$$\begin{array}{rcl} \pi_1 : B' & \longrightarrow & \mathbb{C}^*/\mathbb{Z}_2 \\ (\lambda, \mu) & \mapsto & \lambda \\ \pi_2 : B' & \longrightarrow & \mathbb{C}^*/\mathbb{Z}_2 \\ (\lambda, \mu) & \mapsto & \mu \\ \pi_3 : B' & \longrightarrow & \mathbb{C}^*/\mathbb{Z}_2 \\ (\lambda, \mu) & \mapsto & \lambda^{-1}\mu^{-1} \end{array} \tag{4.10}$$

($\mathbb{Z}_2$ acts on $\mathbb{C}^*$ by $x \sim x^{-1}$), we can obtain three pullback bundles

$$\begin{array}{ccc} \overline{Z}'_i & \longrightarrow & \overline{X}_4/\mathbb{Z}_2 \\ \downarrow & & \downarrow \\ B' & \xrightarrow{\pi_i} & (\mathbb{C}^* - \{\pm 1\})/\mathbb{Z}_2 \end{array}$$

$i = 1, 2, 3$, such that $\overline{Z}' \cong \overline{Z}'_1 \times \overline{Z}'_2 \times \overline{Z}'_3 \cong \pi_1^*(\overline{X}_4/\mathbb{Z}_2) \times \pi_2^*(\overline{X}_4/\mathbb{Z}_2) \times f^*\pi_3^*(\overline{X}_4/\mathbb{Z}_2)$, with $f(x) = -x$.

As a consequence, if we write $R(\overline{X}_4/\mathbb{Z}_2) = aT + bS_2 + cS_{-2} + dS_0$, as in (3.3), the Hodge monodromy representation is

$$\begin{aligned}
R(\overline{Z}') =&\pi_1^*(R(\overline{X}_4/\mathbb{Z}_2)) \otimes \pi_2^*(R(\overline{X}_4/\mathbb{Z}_2)) \otimes f^*\pi_3^*(R(\overline{X}_4/\mathbb{Z}_2)) \qquad (4.11)\\
=&(aT + bS_2^\lambda + cS_{-2}^\lambda + dS_0^\lambda) \otimes (aT + bS_2^\mu + cS_{-2}^\mu + dS_0^\mu) \otimes (aT + cS_2^{\lambda\mu} + bS_{-2}^{\lambda\mu} + dS_0^{\lambda\mu})\\
=&a^3T + a^2bS_2^\mu + a^2cS_{-2}^\mu + a^2dS_0^\mu + a^2bS_2^\lambda + ab^2S_2^\lambda \otimes S_2^\mu + abcS_2^\lambda \otimes S_{-2}^\mu + abdS_2^\lambda \otimes S_0^\mu\\
&+ a^2cS_{-2}^\lambda + abcS_{-2}^\lambda \otimes S_2^\mu + ac^2S_{-2}^\lambda \otimes S_{-2}^\mu + acdS_{-2}^\lambda \otimes S_0^\mu + a^2dS_0^\lambda + abdS_0^\lambda \otimes S_2^\mu\\
&+ acdS_0^\lambda \otimes S_{-2}^\mu + ad^2S_0^\lambda \otimes S_0^\mu + a^2cS_2^{\lambda\mu} + abcS_2^\mu \otimes S_2^{\lambda\mu} + ac^2S_{-2}^\mu \otimes S_2^{\lambda\mu} + acdS_0^\mu \otimes S_2^{\lambda\mu}\\
&+ abcS_2^\lambda \otimes S_2^{\lambda\mu} + b^2cS_2^\lambda \otimes S_2^\mu \otimes S_2^{\lambda\mu} + bc^2S_2^\lambda \otimes S_{-2}^\mu \otimes S_2^{\lambda\mu} + bcdS_2^\lambda \otimes S_0^\mu \otimes S_2^{\lambda\mu}\\
&+ ac^2S_{-2}^\lambda \otimes S_2^{\lambda\mu} + bc^2S_{-2}^\lambda \otimes S_2^\mu \otimes S_2^{\lambda\mu} + c^3S_{-2}^\lambda \otimes S_{-2}^\mu \otimes S_2^{\lambda\mu} + c^2dS_{-2}^\lambda \otimes S_0^\mu \otimes S_2^{\lambda\mu}\\
&+ acdS_0^\lambda \otimes S_2^{\lambda\mu} + bcdS_0^\lambda \otimes S_2^\mu \otimes S_2^{\lambda\mu} + c^2dS_0^\lambda \otimes S_{-2}^\mu \otimes S_2^{\lambda\mu} + cd^2S_0^\lambda \otimes S_0^\mu \otimes S_2^{\lambda\mu}\\
&+ a^2bS_{-2}^{\lambda\mu} + ab^2S_2^\mu \otimes S_{-2}^{\lambda\mu} + abcS_{-2}^\mu \otimes S_{-2}^{\lambda\mu} + abdS_0^\mu \otimes S_{-2}^{\lambda\mu}\\
&+ ab^2S_2^\lambda \otimes S_{-2}^{\lambda\mu} + b^3S_2^\lambda \otimes S_2^\mu \otimes S_{-2}^{\lambda\mu} + b^2cS_2^\lambda \otimes S_{-2}^\mu \otimes S_{-2}^{\lambda\mu} + b^2dS_2^\lambda \otimes S_0^\mu \otimes S_{-2}^{\lambda\mu}\\
&+ abcS_{-2}^\lambda \otimes S_{-2}^{\lambda\mu} + b^2cS_{-2}^\lambda \otimes S_2^\mu \otimes S_{-2}^{\lambda\mu} + bc^2S_{-2}^\lambda \otimes S_{-2}^\mu \otimes S_{-2}^{\lambda\mu} + bcdS_{-2}^\lambda \otimes S_0^\mu \otimes S_{-2}^{\lambda\mu}\\
&+ abdS_0^\lambda \otimes S_{-2}^{\lambda\mu} + b^2dS_0^\lambda \otimes S_2^\mu \otimes S_{-2}^{\lambda\mu} + bcdS_0^\lambda \otimes S_{-2}^\mu \otimes S_{-2}^{\lambda\mu} + bd^2S_0^\lambda \otimes S_0^\mu \otimes S_{-2}^{\lambda\mu}\\
&+ a^2dS_0^{\lambda\mu} + abdS_2^\mu \otimes S_0^{\lambda\mu} + acdS_{-2}^\mu \otimes S_0^{\lambda\mu} + ad^2S_0^\mu \otimes S_0^{\lambda\mu}\\
&+ abdS_2^\lambda \otimes S_0^{\lambda\mu} + b^2dS_2^\lambda \otimes S_2^\mu \otimes S_0^{\lambda\mu} + bcdS_2^\lambda \otimes S_{-2}^\mu \otimes S_0^{\lambda\mu} + bd^2S_2^\lambda \otimes S_0^\mu \otimes S_0^{\lambda\mu}\\
&+ acdS_{-2}^\lambda \otimes S_0^{\lambda\mu} + bcdS_{-2}^\lambda \otimes S_2^\mu \otimes S_0^{\lambda\mu} + c^2dS_{-2}^\lambda \otimes S_{-2}^\mu \otimes S_0^{\lambda\mu} + cd^2S_{-2}^\lambda \otimes S_0^\mu \otimes S_0^{\lambda\mu}\\
&+ ad^2S_0^\lambda \otimes S_0^{\lambda\mu} + bd^2S_0^\lambda \otimes S_2^\mu \otimes S_0^{\lambda\mu} + cd^2S_0^\lambda \otimes S_{-2}^\mu \otimes S_0^{\lambda\mu} + d^3S_0^\lambda \otimes S_0^\mu \otimes S_0^{\lambda\mu},
\end{aligned}$$

where $S_\bullet^\lambda = \pi_1^*(S_\bullet), S_\bullet^\mu = \pi_2^*(S_\bullet)$ and $S_\bullet^{\lambda\mu} = \pi_3^*(S_\bullet)$. To obtain the E-polynomial of the total space, we need to substitute each representation by its associated E-polynomial, by Theorem 2.3.2.

Proposition 4.2.2. *We have*

- $e(T) = e(S_2^\lambda \otimes S_2^\mu \otimes S_2^{\lambda\mu}) = e(S_2^\lambda \otimes S_{-2}^\mu \otimes S_{-2}^{\lambda\mu}) = q^2 - 6q + 9,$
- $e(S_0^\bullet) = e(S_{-2}^\lambda \otimes S_{-2}^\mu \otimes S_{-2}^{\lambda\mu}) = e(S_{-2}^\lambda \otimes S_2^\mu \otimes S_2^{\lambda\mu}) = -2q + 6,$
- $e(S_{\pm 2}^\bullet) = e(S_{\pm 2}^\bullet \otimes S_{\pm 2}^\bullet) = -q + 5,$

for $\bullet = \lambda, \mu, \lambda\mu$.

Proof. Recall that the basis is $B' = \{(\lambda, \mu) \in (\mathbb{C}^*)^2 | \lambda, \mu, \lambda\mu \neq \pm 1\}/\mathbb{Z}_2$. We compute the E-polynomial of each representation case by case:

- $e(T)$. Using (2.3.8), we compute $e((\mathbb{C}-\{0,\pm 1\})^2/\mathbb{Z}_2) = (q-2)^2+1$, since $e(\mathbb{C}-\{0,\pm 1\})^+ = q-2$ and $e(\mathbb{C}-\{0,\pm 1\})^- = -1$. Also $e(\{(\lambda,\mu)\in(\mathbb{C}-\{0,\pm 1\})^2|\lambda\mu=\pm 1\}/\mathbb{Z}_2) = 2e((\mathbb{C}-\{0,\pm 1\})/\mathbb{Z}_2) = 2(q-2)$. So $e(T)=e(B') = q^2-4q+5-(2q-4) = q^2-6q+9$.

- $e(S_0^\lambda)$. Since $S_0^\lambda = \pi_1^*(S_0)$, we need to compute the E-polynomial of the pullback bundle of the fibration $\mathbb{C}-\{0,\pm 1\}\longrightarrow\mathbb{C}-\{\pm 2\}$ that maps $\lambda\mapsto\lambda+\lambda^{-1}$, and which has Hodge monodromy representation equal to $T+S_0$. The pullback bundle is

$$\begin{tikzcd}
E_{S_0}\subset \bar{E}_{S_0} = (\mathbb{C}-\{0,\pm 1\})\times(\mathbb{C}-\{0,\pm 1\}) \arrow[r] \arrow[d, "p"] & \mathbb{C}-\{0,\pm 1\} \arrow[d, "g"] \\
B'\subset \bar{B}' = (\mathbb{C}-\{0,\pm 1\})\times(\mathbb{C}-\{0,\pm 1\})/\mathbb{Z}_2 \arrow[r, "\pi_1"] & \mathbb{C}-\{\pm 2\}\cong(\mathbb{C}-\{0,\pm 1\})/\mathbb{Z}_2
\end{tikzcd}$$

where $g(\lambda)=\lambda+\lambda^{-1}$ and p is the quotient map. $E_{S_0}=p^{-1}(B')$, and $B'=\bar{B}'-\{(\lambda,\mu)|\lambda\mu=\pm 1\}$. Then $e(\bar{E}_{S_0}) = (q-3)^2$, $e(p^{-1}(\{(\lambda,\mu)|\lambda\mu=\pm 1\})) = e(\{(\lambda,\mu)\in(\mathbb{C}-\{0,\pm 1\})^2|\lambda\mu=\pm 1\}) = 2(q-3)$. So $e(E_{S_0}) = e(T+S_0^\lambda) = (q-3)^2-2(q-3) = q^2-8q+15$. Finally,

$$e(S_0^\lambda) = e(T+S_0^\lambda)-e(T) = -2q+6.$$

The diagram still commutes if we change π_1 by π_2 or π_3, so we infer that $S_0^\lambda\cong S_0^\mu\cong S_0^{\lambda\mu}$ as local systems. This implies that certain reductions can be made: since $S_2^\bullet\otimes S_{-2}^\bullet\cong S_0^\bullet$, whenever S_0^λ, S_0^μ or $S_0^{\lambda\mu}$ appear in a tensor product we can switch one by another in order to simplify the expression. For example:

$$S_2^\lambda\otimes S_2^\mu\otimes S_0^{\lambda\mu}\cong S_2^\lambda\otimes S_2^\mu\otimes S_0^\mu\cong S_2^\lambda\otimes S_{-2}^\mu,$$

$$S_0^\lambda\otimes S_{-2}^\mu\otimes S_0^{\lambda\mu}\cong S_0^\mu\otimes S_{-2}^\mu\otimes S_0^\mu\cong S_{-2}^\mu,$$

and so on. In fact, we deduce from this that, for all representations R that appear in (4.11), either $e(R)=e(S_0^\lambda)$, $e(R)=e(S_{\pm 2}^\lambda)$, $e(R)=e(S_{\pm 2}^\lambda\otimes S_{\pm 2}^\mu)$ or $e(R) = e(S_{\pm 2}^\lambda\otimes S_{\pm 2}^\mu\otimes S_{\pm 2}^{\lambda\mu})$, as any representation where $S_0^\bullet$ appears can be simplified to one of these expressions.

- $e(S_{-2}^\lambda)$. To obtain $e(S_{-2}^\lambda)$, we take the pullback under the map $\mathbb{C}-\{\pm 2,0\}\to\mathbb{C}-\{\pm 2\}$, $x\mapsto x^2-2$, which ramifies at -2. The pullback fibration is

$$\begin{tikzcd}
E_{S_{-2}}\subset \bar{E}_{S_{-2}} = \left\{\frac{(y,\mu)}{(y,\mu)\sim(y^{-1},\mu^{-1})}\right\} \arrow[r] \arrow[d, "p"] & x=y+y^{-1}\in\mathbb{C}-\{\pm 2,0\} \arrow[d, "2:1"] \\
B'\subset \bar{B}' = \{[(\lambda=y^2,\mu)]\} \arrow[r] & x^2-2=y^2+y^{-2}=\lambda+\lambda^{-1}\in\mathbb{C}-\{\pm 2\}
\end{tikzcd}$$

where $p(y,\mu) = (y^2,\mu) \in B'$, $y \in \mathbb{C} - \{0, \pm 1, \pm\sqrt{-1}\}$. Therefore $e(\bar{E}_{S_{-2}}) = (q-3)(q-2)+2$. Substracting the contribution corresponding to the two hyperbolas $\{y^2\mu = \pm 1\}$, which has E-polynomial $2(q-3)$, we get $e(E_{S_{-2}}) = e(T + S^{\lambda}_{-2}) = (q-3)(q-2)+2-2(q-3) = q^2-7q+14$, and

$$e(S^{\lambda}_{-2}) = e(T + S^{\lambda}_{-2}) - e(T) = -q+5.$$

The diagram for $e(S^{\lambda}_{2})$ is similar:

$$\begin{tikzcd}
E_{S_2} \subset \bar{E}_{S_2} = \left\{ \frac{(y,\mu)}{(y,\mu)\sim(y^{-1},\mu^{-1})} \right\} \arrow[r] \arrow[d, "p"] & x = y + y^{-1} \in \mathbb{C} - \{\pm 2, 0\} \arrow[d, "2:1"] \\
B' \subset \bar{B}' = \{[(\lambda = -y^2, \mu)]\} \arrow[r] & 2 - x^2 = y^2 + y^{-2} = \lambda + \lambda^{-1} \in \mathbb{C} - \{\pm 2\}
\end{tikzcd}$$

but now $p(y,\mu) = (-y^2,\mu)$. Since $E_{S_2} \cong E_{S_{-2}}$, we obtain that $e(S^{\lambda}_{2}) = e(S^{\lambda}_{-2})$ and analogous computations yield that $e(S^{\lambda}_{\pm 2}) = e(S^{\mu}_{\pm 2}) = e(S^{\lambda\mu}_{\pm 2})$.

- $e(S^{\lambda}_{-2} \otimes S^{\mu}_{-2})$. To compute the E-polynomial of this representation, we can use the fibration with Hodge monodromy representation $(T + S^{\lambda}_{-2}) \otimes (T + S^{\mu}_{-2})$, which corresponds to the fibered product of two copies of the pullback fibration $E_{S_{-2}}$, one with $\lambda = y^2$, the other with $\mu = z^2$. The total space is

$$E_{S_{-2,-2}} \subset \bar{E}_{S_{-2,-2}} = \left\{ \left(\frac{(y,\mu)}{(y,\mu)\sim(y^{-1},\mu^{-1})}, \frac{(\lambda,z)}{(\lambda,z)\sim(\lambda^{-1},z^{-1})} \right) \right\} \to B' \subset \bar{B}',$$

where $\bar{B}' = \{[\lambda,\mu]\}$, $\lambda = y^2, \mu = z^2$ and $(y,z) \sim (y^{-1}, z^{-1}), y, z \in \mathbb{C} - \{0, \pm 1, \pm\sqrt{-1}\}$. The E-polynomial is $e(\bar{E}_{S_{-2,-2}}) = (q-3)^2 + 4$. Recall that we need to substract the contribution of the (now four) hyperbolas given by $\{yz = \pm 1\}$ and $\{yz = \pm\sqrt{-1}\}$. The action $(y,z) \sim (y^{-1}, z^{-1})$ acts on each of the first two hyperbolas giving a contribution of $2(q-3)$, whereas it interchanges the last two, which gives $q-5$. We get $e((T + S^{\lambda}_{-2}) \otimes (T + S^{\mu}_{-2})) = e(E_{S_{-2,-2}}) = (q-3)^2 + 4 - 2(q-3) - (q-5) = q^2 - 9q + 24$ and

$$\begin{aligned} e(S^{\lambda}_{-2} \otimes S^{\mu}_{-2}) &= e((T + S^{\lambda}_{-2}) \otimes (T + S^{\mu}_{-2})) - e(T) - e(S^{\lambda}_{-2}) - e(S^{\mu}_{-2}) \\ &= q^2 - 9q + 24 - (q^2 - 6q + 9) - 2(-q+5) = -q+5. \end{aligned}$$

Computing the different fibered products of E_{S_2} and $E_{S_{-2}}$ gives us

$$e(S^{\lambda}_{2} \otimes S^{\mu}_{-2}) = e(S^{\lambda}_{-2} \otimes S^{\mu}_{2}) = e(S^{\lambda}_{2} \otimes S^{\mu}_{2}) = -q+5.$$

Changing the projection does not alter the computations, so $e(S^{\bullet}_{\pm 2} \otimes S^{\bullet}_{\pm 2}) = -q+5$.

- $e(S^{\lambda}_{-2} \otimes S^{\mu}_{-2} \otimes S^{\lambda\mu}_{-2})$. To compute this E-polynomial, we can compute the E-polynomial of the representation $(T+S^{\lambda}_{2}) \otimes (T+S^{\mu}_{2}) \otimes (T+S^{\lambda\mu}_{2})$, which corresponds to the fibered product of three copies of $E_{S_{-2}}$. The total space of this fibered product is given by

$$\left(\frac{(y,\mu)}{(y,\mu)\sim(y^{-1},\mu^{-1})}, \frac{(\lambda,z)}{(\lambda,z)\sim(\lambda^{-1},z^{-1})}, \frac{(w,\lambda^{-1})}{(w,\lambda^{-1})\sim(w^{-1},\lambda)} \right),$$

where $\lambda = y^2, \mu = z^2, \lambda\mu = w^2$ and $yz = \pm w$, $y,z,w \in \mathbb{C} - \{0, \pm 1, \pm\sqrt{-1}\}$ and $(y,z) \sim (y^{-1}, z^{-1})$. Taking into account the two possible signs for w, and substracting the contribution from the hyperbolas in $\bar{B}'$, we obtain that the E-polynomial is $2(q^2 - 9q + 24)$. This implies that

$$\begin{aligned} e(S^{\lambda}_{-2} \otimes S^{\mu}_{-2} \otimes S^{\lambda\mu}_{-2}) &= e((T+S^{\lambda}_{-2}) \otimes (T+S^{\mu}_{-2}) \otimes (T+S^{\lambda\mu}_{-2})) \\ &\quad - e(T) - 3e(S^{\lambda}_{-2}) - 3e(S^{\lambda}_{-2} \otimes S^{\mu}_{-2}) \\ &= 2(q^2 - 9q + 24) - (q^2 - 6q + 9) - 6(-q+5) \\ &= q^2 - 6q + 9 = e(T). \end{aligned}$$

An analogous computation gives the same polynomial for $e(S^{\lambda}_{2} \otimes S^{\mu}_{2} \otimes S^{\lambda\mu}_{-2})$ and cyclic permutations of signs.

- $e(S^{\lambda}_{-2} \otimes S^{\lambda}_{-2} \otimes S^{\lambda}_{2})$. As we did in the previous case, to compute the E-polynomial it suffices to take the fibered product of two copies of $E_{S_{-2}}$ and E_{S_2}. The total space is again parametrized by

$$\left(\frac{(y,\mu)}{(y,\mu)\sim(y^{-1},\mu^{-1})}, \frac{(\lambda,z)}{(\lambda,z)\sim(\lambda^{-1},z^{-1})}, \frac{(w,\lambda^{-1})}{(w,\lambda^{-1})\sim(w^{-1},\lambda)} \right),$$

where now $\lambda = y^2, \mu = z^2$ and $\lambda\mu = -w^2, y, z, w \in \mathbb{C} - \{0, \pm 1, \pm\sqrt{-1}\}$. In particular, this implies that $yz = \sqrt{-1}w$ and $yz = -\sqrt{-1}w$, which gives two components that get identified under the $\mathbb{Z}_2$-action given by $(y,z,w) \sim (y^{-1}, z^{-1}, w^{-1})$. Therefore, the quotient is parametrized by $(y,z) \in (\mathbb{C} - \{0, \pm 1, \pm\sqrt{-1}\})^2$, which produces $(q-5)^2$. Substracting the four hyperbolas, we get that the E-polynomial of the fibered product is $(q-5)^2 - 4(q-5) = q^2 - 14q + 45$. So

$$\begin{aligned} e(S^{\lambda}_{-2} \otimes S^{\mu}_{-2} \otimes S^{\lambda\mu}_{2}) &= e((T+S^{\lambda}_{-2}) \otimes (T+S^{\mu}_{-2}) \otimes (T+S^{\lambda\mu}_{2})) \\ &\quad - e(T) - 3e(S^{\lambda}_{-2}) - 3e(S^{\lambda}_{-2} \otimes S^{\mu}_{-2}) \\ &= (q^2 - 14q + 45) - (q^2 - 6q + 9) - 6(-q+5) \\ &= -2(q-3) = e(S^{\bullet}_{0}). \end{aligned}$$

□

Substituting the values just obtained for the E-polynomial of every irreducible representation in (4.11), and the values $a = q^3, b = -3q, c = 3q^2, d = -1$, we obtain the E-polynomial of the total fibration

$$\begin{aligned} e(\overline{Z}/\mathbb{Z}_2) &= e(\overline{Z}') = e(R(\overline{Z}')) \\ &= q^{11} - 6q^{10} + 54q^8 - 12q^7 + 189q^6 - 915q^5 + 666q^4 + 153q^3 - 81q^2 - 43q - 6. \end{aligned} \tag{4.12}$$

Using the formula in Proposition 2.4.3, we get that

$$\begin{aligned} e(W_5) = e(Z) &= (q^2 - q)e(\overline{Z}') + q\, e(\overline{Z}) \\ &= q^{13} - 6q^{12} - 2q^{11} + 51q^{10} + 12q^9 + 216q^8 - 573q^7 \\ &\quad - 198q^6 + 696q^5 - 18q^4 - 125q^3 - 45q^2 - 9q. \end{aligned}$$

4.2.6 Generic case

Let us finally deal with the case $(t_1, t_2, t_3) \notin C$, $t_i \neq \pm 2$, which corresponds to the open subset U in $(\mathbb{C}^*)^3$ defined by those representations whose ξ_1, ξ_2, ξ_3 are diagonalizable and do not share an eigenvector. Choosing a basis that diagonalizes ξ_1, note that, if we write $\xi_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \xi_3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

so that

$$\xi_3 = \begin{pmatrix} -\lambda_1^{-1} d & \lambda_1 b \\ \lambda_1^{-1} c & -\lambda_1 a \end{pmatrix}$$

and $bc \neq 0$ (see Lemma 4.2.1). Conjugating by a diagonal matrix, we can assume that $b = 1$. As $t_2 = a + d$ and $t_3 = -\lambda_1 a - \lambda_1^{-1} d$, we have that a, d are determined by the values of (t_2, t_3); and c is determined by the equation $\det \xi_2 = ad - bc = 1$. We see that for fixed (λ_1, t_2, t_3), a, b, c, d are fully determined, and so are ξ_2, ξ_3.

Consider the $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$-cover given by $(\lambda_1, \lambda_2, \lambda_3) \mapsto (t_1, t_2, t_3)$ over $(\mathbb{C} - \{0, \pm 1\})^3$. Let $\bar{E}$ be the pull-back fibration. The fiber over $(\lambda_1, \lambda_2, \lambda_3)$ is isomorphic to

$$\overline{X}_{4,\lambda_1} \times \overline{X}_{4,\lambda_2} \times \overline{X}_{4,\lambda_3}.$$

Let $(A_1, B_1, A_2, B_2, A_3, B_3) \in \bar{E}$. Then $[A_1, B_1] = \xi_1$, $[A_2, B_2] = \xi_2$, $[A_3, B_3] = \xi_3$, where $a = a(\lambda_1, \lambda_2, \lambda_3), b = 1, c = c(\lambda_1, \lambda_2, \lambda_3), d = d(\lambda_1, \lambda_2, \lambda_3)$. Take $Q = Q(\lambda_1, \lambda_2, \lambda_3)$, $S = S(\lambda_1, \lambda_2, \lambda_3)$ matrices such that $Q^{-1}\xi_2 Q = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2^{-1} \end{pmatrix}$ and $S^{-1}\xi_3 S = \begin{pmatrix} \lambda_3 & 0 \\ 0 & \lambda_3^{-1} \end{pmatrix}$. Then $\Theta(A_1, B_1, A_2, B_2, A_3, B_3) = (A_1, B_1, Q^{-1}A_2 Q, Q^{-1}B_2 Q, S^{-1}A_3 S, S^{-1}B_3 S)$ identifies $\bar{E}$ with the subset of $\overline{X}_4 \times \overline{X}_4 \times \overline{X}_4$ where $(t_1, t_2, t_3) \notin C$. The second and third copies of $\mathbb{Z}_2$ act as the standard $\mathbb{Z}_2$-action on the second and third copies of $\overline{X}_4$, respectively. The action of the first copy of $\mathbb{Z}_2$ is more delicate: it acts as conjugation by $P_0 = \begin{pmatrix} 0 & 1 \\ c & 0 \end{pmatrix}$, sending $\xi_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix} \mapsto \begin{pmatrix} \lambda_1^{-1} & 0 \\ 0 & \lambda_1 \end{pmatrix}$, $\xi_2 = \begin{pmatrix} a & 1 \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & 1 \\ c & a \end{pmatrix}$. Then under the isomorphism Θ, it acts by conjugation by $Q^{-1}P_0Q$ (resp. $S^{-1}P_0S$) on the second (resp. third) factor of $\overline{X}_4$. This matrix is easily computed to be diagonal, so the action is (homologically) trivial.

The conclusion is that $\bar{E}/\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is isomorphic to the open set of $\overline{X}_4/\mathbb{Z}_2 \times \overline{X}_4/\mathbb{Z}_2 \times \overline{X}_4/\mathbb{Z}_2$, where $(t_1, t_2, t_3) \notin C$.

Now we need to compute the E-polynomial of $T = \left(\overline{X}_4/\mathbb{Z}_2 \times \overline{X}_4/\mathbb{Z}_2 \times \overline{X}_4/\mathbb{Z}_2\right) \cap \{(t_1, t_2, t_3) \in C\}$. Here we can parametrize $C \cong \{(\lambda, \mu) \in (\mathbb{C}^*)^3 | \lambda, \mu, \lambda\mu \neq \pm 1\}/\mathbb{Z}_2$. The fiber over (λ, μ) is isomorphic to $\overline{X}_{4,\lambda} \times \overline{X}_{4,\mu} \times \overline{X}_{4,-\lambda^{-1}\mu^{-1}}$, hence this space is isomorphic to $\overline{Z}'$, studied in (4.9). Using (4.12), we get

$$\begin{aligned} e(W_6) &= e(\mathrm{PGL}(2,\mathbb{C}))(e(\overline{X}_4/\mathbb{Z}_2)^3 - e(\overline{Z}')) \\ &= q^{15} - 7q^{14} + 8q^{13} + 44q^{12} - 78q^{11} - 136q^{10} - 153q^9 \\ &\quad + 1149q^8 - 450q^7 - 1263q^6 + 798q^5 + 265q^4 - 119q^3 - 52q^2 - 7q. \end{aligned}$$

4.2.7 Final result

If we add all the E-polynomials of the different strata, we get

$$\begin{aligned} e(W) &= e(W_1) + e(W_2) + e(W_3) + e(W_4) + e(W_5) + e(W_6) \\ &= q^{15} - 5q^{13} + 10q^{11} - 252q^{10} - 20q^9 + 20q^7 + 252q^6 - 10q^5 + 5q^3 - q. \end{aligned}$$

Then

$$e(\mathcal{M}^{g=3}_{-\mathrm{Id}}) = e(W)/e(\mathrm{PGL}(2,\mathbb{C})) = q^{12} - 4q^{10} + 6q^8 - 252q^7 - 14q^6 - 252q^5 + 6q^4 - 4q^2 + 1.$$

This agrees with the result in [60], obtained by arithmetic methods.

4.3 Hodge monodromy representation for the genus 2 character variety

We introduce the following sets associated to the representations of a genus 2 complex curve, and give the E-polynomials computed in [53]:

- $Y_0 := \{(A_1, B_1, A_2, B_2) \in \mathrm{SL}(2,\mathbb{C})^4 \,|\, [A_1, B_1][A_2, B_2] = \mathrm{Id}\}$. Then $e(Y_0) = q^9 + q^8 + 12q^7 + 2q^6 - 3q^4 - 12q^3 - q$, by [53, Section 8.1].

- $Y_1 := \{(A_1, B_1, A_2, B_2) \in \mathrm{SL}(2,\mathbb{C})^4 \,|\, [A_1, B_1][A_2, B_2] = -\mathrm{Id}\}$. Then $e(Y_1) = q^9 - 3q^7 - 30q^6 + 30q^4 + 3q^3 - q$, by [53, Section 9].

- $\overline{Y}_2 := \left\{ (A_1, B_1, A_2, B_2) \in \mathrm{SL}(2,\mathbb{C})^4 \mid [A_1,B_1][A_2,B_2] = J_+ = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right) \right\}$, $e(\overline{Y}_2) = q^9 - 3q^7 - 4q^6 - 39q^5 - 4q^4 - 15q^3$, by [53, Section 11].
- $\overline{Y}_3 := \left\{ (A_1, B_1, A_2, B_2) \in \mathrm{SL}(2,\mathbb{C})^4 \mid [A_1,B_1][A_2,B_2] = J_- = \left(\begin{smallmatrix} -1 & 1 \\ 0 & -1 \end{smallmatrix} \right) \right\}$. Then $e(\overline{Y}_3) = q^9 - 3q^7 + 15q^6 + 6q^5 + 45q^4$, by [53, Section 12].
- $\overline{Y}_{4,\lambda} := \left\{ (A_1, B_1, A_2, B_2) \mid [A_1,B_1][A_2,B_2] = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix} \right) \right\}$, for $\lambda \neq 0, \pm 1$. Then $e(\overline{Y}_{4,\lambda}) = q^9 - 3q^7 + 15q^6 - 39q^5 + 39q^4 - 15q^3 + 3q^2 - 1$, by [53, Section 10].

Let

$$\overline{Y}_4 := \left\{ (A_1, B_1, A_2, B_2, \lambda) \in \mathrm{SL}(2,\mathbb{C})^4 \times \mathbb{C}^* \mid [A_1,B_1][A_2,B_2] = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix} \right), \lambda \neq 0, \pm 1 \right\}.$$

We have a fibration

$$\overline{Y}_4 \longrightarrow \mathbb{C} - \{0, \pm 1\}.$$

If we take the quotient by the $\mathbb{Z}_2$-action there is another fibration

$$\overline{Y}_4/\mathbb{Z}_2 \longrightarrow \mathbb{C} - \{\pm 2\}.$$

We are interested in the Hodge monodromy representations $R(\overline{Y}_4)$ and $R(\overline{Y}_4/\mathbb{Z}_2)$.

Proposition 4.3.1. *$R(\overline{Y}_4) = (q^9 - 3q^7 + 6q^5 - 6q^4 + 3q^2 - 1)T + (15q^6 - 45q^5 + 45q^4 - 15q^3)N$.*

Proof. We follow the stratification $\overline{Y}_{4,\lambda_0} = \bigsqcup_{i=1}^{7} Z_i$ given in [53, Section 10], and study the behaviour of each stratum when λ varies in $\mathbb{C} - \{0, \pm 1\}$ to obtain the Hodge monodromy representation of $\overline{Y}_4$. Let $\xi = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$. As in [53, Section 10], we write

$$\nu = [B_2, A_2] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \delta = [A_1, B_1] = \xi \nu = \begin{pmatrix} \lambda a & \lambda b \\ \lambda^{-1} c & \lambda^{-1} d \end{pmatrix},$$

and $t_1 = \operatorname{tr} \nu$, $t_2 = \operatorname{tr} \delta$. Note that every (t_1, t_2, λ) determines a, d by

$$a = \frac{t_2 - \lambda^{-1} t_1}{\lambda - \lambda^{-1}}, \quad d = \frac{\lambda t_1 - t_2}{\lambda - \lambda^{-1}}.$$

Then $bc = ad - 1$.

We look at the strata:

- Z_1, corresponding to $t_1 = \pm 2, t_2 = \pm 2$. In this case, both ν, δ are of Jordan type. If we take the basis given by $\{u_1, u_2\}$, where u_1 is an eigenvector for ν and u_2 an eigenvector for δ, then
$$\nu = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad \delta = \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}$$

for certain $x, y \in \mathbb{C}^*$. Now, since $\delta\nu^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, we obtain that $\lambda + \lambda^{-1} = 2 - xy$. We can fix $x = 1$ by rescaling the basis, so y is fixed and there is no monodromy around the origin. Therefore

$$\begin{aligned} R(Z_1) &= e(Z_1)T = (q-1)(e(\overline{X}_2) + e(\overline{X}_3))^2 T \\ &= (4q^7 - 15q^5 + 5q^4 + 15q^3 - 9q^2)T, \end{aligned}$$

where T is the trivial representation.

- Z_2, corresponding to $t_1 = 2, t_2 = \lambda + \lambda^{-1}$ and $t_2 = 2, t_1 = \lambda + \lambda^{-1}$. We focus on the first case. In this situation, $bc = 0$, so there are three possibilities: either $b = c = 0$ (in which case $\nu = \mathrm{Id}$) or $b = 0, c \neq 0$ or $b \neq 0, c = 0$ (in either case there is a parameter in $\mathbb{C}^*$ and ν is of Jordan type). In every situation, ν has trivial monodromy, whereas $\delta \sim \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$. This contributes $R(\overline{X}_4)$. Therefore

$$\begin{aligned} R(Z_2) &= 2(e(X_0) + 2(q-1)e(\overline{X}_2))R(\overline{X}_4) \\ &= (6q^7 - 4q^6 - 6q^5 - 2q^4 + 4q^3 + 6q^2 - 4q)T \\ &+ (18q^6 - 30q^5 - 6q^4 + 30q^3 - 12q^2)N. \end{aligned}$$

- Z_3, given by $t_1 = -2, t_2 = -\lambda - \lambda^{-1}$ and $t_2 = -2, t_1 = -\lambda - \lambda^{-1}$. This is analogous to the previous case, so

$$\begin{aligned} R(Z_3) &= 2(e(X_1) + 2(q-1)e(\overline{X}_3))R(\overline{X}_4) \\ &= (4q^7 + 10q^6 - 12q^5 - 6q^4 - 10q^3 + 12q^2 + 2q)T \\ &+ (12q^6 + 18q^5 - 66q^4 + 30q^3 + 6q^2)N. \end{aligned}$$

- Z_4, defined by $t_1 = 2, t_2 \neq \pm 2, \lambda + \lambda^{-1}$ and $t_2 = 2, t_1 \neq \pm 2, \lambda + \lambda^{-1}$. Both cases are similar, so we do the first case. For each λ, (t_1, t_2) move in a punctured line $\{(t_1, t_2) \mid t_1 = 2, t_2 \neq \pm 2, \lambda + \lambda^{-1}\}$, where ν is of Jordan form and δ is of diagonal type, with trace t_2. Both families can be trivialized, giving a contribution of $e(\overline{X}_2)$ and $e(\overline{X}_4/\mathbb{Z}_2)$. The missing fiber $\overline{X}_{4,\lambda}$ over $\lambda + \lambda^{-1}$, which needs to be removed, has monodromy representation $R(\overline{X}_4)$ as λ varies. Therefore

$$\begin{aligned} R(Z_4) &= 2(q-1)e(\overline{X}_2)(e(\overline{X}_4/\mathbb{Z}_2)T - R(\overline{X}_4)) \\ &= (2q^8 - 12q^7 + 10q^6 + 36q^5 - 26q^4 - 36q^3 + 14q^2 + 12q)T \\ &+ (-6q^6 + 24q^5 - 12q^4 - 24q^3 + 18q^2)N. \end{aligned}$$

The factor $(q-1)$ corresponds to the fact that now $bc \neq 0$, so there is the extra freedom given by $\mathbb{C}^*$.

- Z_5, defined by $t_1 = -2, t_2 \neq \pm 2, -\lambda - \lambda^{-1}$ and $t_2 = -2, t_1 \neq \pm 2, -\lambda - \lambda^{-1}$. Similarly to Z_4, we obtain

$$\begin{aligned} R(Z_5) &= 2(q-1)e(\overline{X}_3)(e(\overline{X}_4/\mathbb{Z}_2)T - R(\overline{X}_4)) \\ &= (2q^8 - 2q^7 - 24q^6 + 12q^5 + 34q^4 - 10q^3 - 12q^2)T \\ &+ (-6q^6 - 6q^5 + 30q^4 - 18q^3)N. \end{aligned}$$

- Z_6. This stratum corresponds to the set $\{(t_1, t_2) \mid t_1, t_2 \neq \pm 2, ad = 1\}$, which is a hyperbola H_λ for every λ. Since $bc = 0$, we get a contribution of $2q - 1$, arising from the disjoint cases $b = c = 0$; $b = 0, c \neq 0$; and $b \neq 0, c = 0$. Parametrizing H_λ by a parameter $\mu \in \mathbb{C}^* - \{\pm 1, \pm\lambda^{-1}\}$ as in [53, Section 10], we obtain a fibration over $\mathbb{C}^* - \{\pm 1, \pm\lambda^{-1}\}$ whose fiber over μ is $\overline{X}_{4,\mu} \times \overline{X}_{4,\lambda\mu}$, for fixed λ. When λ varies over $\mathbb{C}^* - \{\pm 1\}$, note that we can extend the local system trivially to the cases $\lambda, \mu = \pm 1$. This extension can be regarded as a local system over the set of $(\lambda, \mu) \in \mathbb{C}^* \times \mathbb{C}^*$

$$\overline{Z}_6 = \overline{X}_4 \times m^*\overline{X}_4 \longrightarrow \mathbb{C}^* \times \mathbb{C}^*,$$

where $m : \mathbb{C}^* \times \mathbb{C}^* \to \mathbb{C}^*$ maps $(\lambda, \mu) \mapsto \lambda\mu$. The Hodge monodromy representation of $\overline{Z}_6$ belongs to $R(\mathbb{Z}_2 \times \mathbb{Z}_2)[q]$ (with generators N_1, N_2 denoting the representation which is not trivial over the generator of the fundamental group of the first and second copy of $\mathbb{C}^*$ respectively, and $N_{12} = N_1 \otimes N_2$). Since $R(\overline{X}_4) = (q^3-1)T + (3q^2-3q)N$, we get

$$\begin{aligned} R_{\mathbb{C}^*\times\mathbb{C}^*}(\overline{Z}_6) &= ((q^3-1)T + (3q^2-3q)N_2) \otimes ((q^3-1)T + (3q^2-3q)N_{12}) \\ &= (q^3-1)^2T + (3q^2-3q)^2N_1 + (3q^2-3q)(q^3-1)N_2 \\ &+ (3q^2-3q)(q^3-1)N_{12}. \end{aligned}$$

We write this as $R_{\mathbb{C}^*\times\mathbb{C}^*}(\overline{Z}_6) = aT + bN_1 + cN_2 + dN_{12}$. To obtain the Hodge monodromy representation over $\lambda \in \mathbb{C}^*$, we use the projection $\pi_1 : \mathbb{C}^* \times \mathbb{C}^* \to \mathbb{C}^*$, $(\lambda, \mu) \mapsto \lambda$. Then $T \mapsto e(T)T$, $N_2 \mapsto e(N_2)T$, $N_1 \mapsto e(T)N$, $N_{12} \mapsto e(N_2)N$ for the representations. Using that $e(T) = q - 1$ and $e(N_2) = 0$ and substracting the contribution from the sets $\mu = \pm 1, \pm\lambda^{-1}$, which yield $4e(\overline{X}_{4,\lambda})R(\overline{X}_4)$, we get

$$\begin{aligned} R(\overline{Z}_6) &= a\,e(T)T + b\,e(T)N + c\,e(N_2)T + d\,e(N_2)N - 4e(\overline{X}_{4,\lambda})R(\overline{X}_4) \\ &= (q^7 - 5q^6 - 12q^5 + 10q^4 + 10q^3 + 12q^2 - 11q - 5)T \\ &+ (-3q^5 - 51q^4 + 99q^3 - 33q^2 - 12q)N \end{aligned}$$

and

$$\begin{aligned}R(Z_6) =&(2q-1)R(\overline{Z}_6)\\ =&(2q^8-11q^7-19q^6+32q^5+10q^4+14q^3-34q^2+q+5)T\\ &+(-6q^6-99q^5+249q^4-165q^3+9q^2+12q)N.\end{aligned}$$

- Z_7, corresponding to the open stratum given by the set of (t_1,t_2) such that $t_i\neq\pm 2$, $i=1,2$ and $(t_1,t_2)\notin H_\lambda$. If we forget about the condition $(t_1,t_2)\in H_\lambda$, Z_7 is a fibration over (t_1,t_2) with fiber isomorphic to $\overline{X}_{4,\mu_1}\times\overline{X}_{4,\mu_2}$, $t_i=\mu_i+\mu_i^{-1}$, $i=1,2$. Its monodromy is trivial, as the local system is trivial when λ varies. The contribution over H_λ, already computed in the previous stratum, is $R(\overline{Z}_6)$. So we get

$$\begin{aligned}R(Z_7) =&(q-1)(e(\overline{X}_4/\mathbb{Z}_2)^2T-R(\overline{Z}_6))\\ =&(q^9-6q^8+8q^7+27q^6-41q^5-21q^4+23q^3+26q^2-11q-6)T\\ &+(3q^6+48q^5-150q^4+132q^3-21q^2-12q)N.\end{aligned}$$

Adding all pieces, we get

$$R(\overline{Y}_4)=(q^9-3q^7+6q^5-6q^4+3q^2-1)T+(15q^6-45q^5+45q^4-15q^3)N.$$

□

Dividing by $q-1$, we get the formula (4.3).

We want to compute the Hodge monodromy representation of $\overline{Y}_4/\mathbb{Z}_2$. We have the following.

Lemma 4.3.2. *The Hodge monodromy representation $R(\overline{Y}_4/\mathbb{Z}_2)$ is of the form $R(\overline{Y}_4/\mathbb{Z}_2)=aT+bS_2+cS_{-2}+dS_0$, for some polynomials $a,b,c,d\in\mathbb{Z}[q]$.*

Proof. The Hodge monodromy representation $R(\overline{Y}_4/\mathbb{Z}_2)$ lies in the representation ring of the fundamental group of $\mathbb{C}-\{\pm 2\}$. Under the double cover $\mathbb{C}-\{0,\pm 1\}\to\mathbb{C}-\{\pm 2\}$, it reduces to $R(\overline{Y}_4)$. By Proposition 4.3.1, $R(\overline{Y}_4)$ is of order 2. Hence $R(\overline{Y}_4/\mathbb{Z}_2)$ has only monodromy of order 2 over the loops $\gamma_{\pm 2}$ around the points ± 2. This is the statement of the lemma. □

To compute $a,b,c,d\in\mathbb{Z}[q]$, we compute the E-polynomial of the twisted $\mathrm{SL}(2,\mathbb{C})$-character variety (4.4) in another way. Stratify

$$W:=\{(A_1,B_1,A_2,B_2,A_3,B_3)\in\mathrm{SL}(2,\mathbb{C})^6\,|\,[A_1,B_1][A_2,B_2]=-[A_3,B_3]\}$$

as follows:

- $W_0 = \{(A_1, B_1, A_2, B_2, A_3, B_3) \mid [A_1, B_1][A_2, B_2] = -[B_3, A_3] = \mathrm{Id}\}$. Then

$$e(W_0) = e(Y_0)e(X_1) = q^{12} + q^{11} + 11q^{10} + q^9 - 12q^8 - 5q^7 - 12q^6 + 3q^5 + 11q^4 + q^2.$$

- $W_1 = \{(A_1, B_1, A_2, B_2, A_3, B_3) \mid [A_1, B_1][A_2, B_2] = -[B_3, A_3] = -\mathrm{Id}\}$. Then

$$\begin{aligned} e(W_1) =& e(Y_1)e(X_0) = q^{13} + 4q^{12} - 4q^{11} - 46q^{10} \\ & - 117q^9 + 72q^8 + 243q^7 - 18q^6 - 124q^5 - 16q^4 + q^3 + 4q^2. \end{aligned}$$

- $W_2 = \{(A_1, B_1, A_2, B_2, A_3, B_3) \mid [A_1, B_1][A_2, B_2] = -[B_3, A_3] \sim J_+\}$. Then

$$\begin{aligned} e(W_2) =& e(\mathrm{PGL}(2,\mathbb{C})/U)e(\overline{Y}_2)e(\overline{X}_3) = q^{14} + 3q^{13} - 4q^{12} - 16q^{11} - 48q^{10} \\ & - 108q^9 + 24q^8 + 76q^7 + 27q^6 + 45q^5. \end{aligned}$$

- $W_3 = \{(A_1, B_1, A_2, B_2, A_3, B_3) \mid [A_1, B_1][A_2, B_2] = -[B_3, A_3] \sim J_-\}$. Then

$$\begin{aligned} e(W_3) =& e(\mathrm{PGL}(2,\mathbb{C})/U)e(\overline{Y}_3)e(\overline{X}_2) = q^{14} - 2q^{13} - 7q^{12} + 23q^{11} - 9q^{10} \\ & - 33q^9 - 93q^8 - 123q^7 + 108q^6 + 135q^5. \end{aligned}$$

- $W_4 = \{(A_1, B_1, A_2, B_2, A_3, B_3) \mid [A_1, B_1][A_2, B_2] = -[B_3, A_3] \sim \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \lambda \neq 0, \pm 1\}$.

- $\overline{W}_4 = \{(A_1, B_1, A_2, B_2, A_3, B_3, \lambda) \mid [A_1, B_1][A_2, B_2] = -[B_3, A_3] = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \lambda \neq 0, \pm 1\}$.

- $\overline{W}_{4,\lambda} = \{(A_1, B_1, A_2, B_2, A_3, B_3) \mid [A_1, B_1][A_2, B_2] = -[B_3, A_3] = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}\}$, where $\lambda \neq 0, \pm 1$.

Using the formula in Section 4.2.7,

$$\begin{aligned} e(W_4) &= e(W) - e(W_0) - e(W_1) - e(W_2) - e(W_3) \\ &= q^{15} - 2q^{14} - 7q^{13} + 6q^{12} + 6q^{11} - 160q^{10} + 237q^9 \\ &\quad + 9q^8 - 171q^7 + 147q^6 - 69q^5 + 5q^4 + 4q^3 - 5q^2 - q. \end{aligned}$$

For the last stratum, note that:

$$\overline{W}_{4,\lambda} = \overline{Y}_{4,\lambda} \times \overline{X}_{4,-\lambda}\,.$$

So $R(\overline{W}_4/\mathbb{Z}_2) = R(\overline{Y}_4/\mathbb{Z}_2) \otimes R(\tau^*\overline{X}_4/\mathbb{Z}_2)$, where $\tau : \mathbb{C} - \{\pm 2\} \to \mathbb{C} - \{\pm 2\}$, $\tau(x) = -x$. Then

$$\begin{aligned}
R(\overline{W}_4/\mathbb{Z}_2) &= R(\overline{Y}_4/\mathbb{Z}_2) \otimes \tau^* R(\overline{X}_4/\mathbb{Z}_2) \\
&= (aT + bS_2 + cS_{-2} + dS_0) \otimes (q^3T + 3q^2S_2 - 3qS_{-2} - S_0) \\
&= (q^3a + 3q^2b - 3qc - d)T + (3q^2a + q^3b - c - 3qd)S_2 \\
&\quad +(-3qa - b + q^3c + 3q^2d)S_{-2} + (-a - 3qb + 3q^2c + q^3d)S_0 \\
&= a'T + b'S_2 + c'S_{-2} + d'S_0
\end{aligned}$$

Using Proposition 2.4.3, we get

$$\begin{aligned}
e(W_4) &= (q^2 - q)e(\overline{W}_4/\mathbb{Z}_2) + q\, e(\overline{W}_4) \\
&= (q^2 - q)((q-2)a' - (b' + c' + d')) + q((q-3)(a' + d') - 2(b' + c')),
\end{aligned}$$

which gives us the equation

$$\begin{aligned}
e(W_4) = {}& a(q^6 - 2q^5 - 4q^4 + 3q^2 + 2q) + b(2q^5 - 7q^4 - 3q^3 + 7q^2 + q)S \\
&+ c(-q^5 - 4q^4 + 4q^2 + q) + d(-5q^4 - q^3 + 5q^2 + q).
\end{aligned} \tag{4.13}$$

We can obtain another equation if we recall that

$$Y_4 := \{(A_1, B_1, A_2, B_2) \mid [A_1, B_1][A_2, B_2] \sim \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \lambda \neq 0, \pm 1\}.$$

Using that $\mathrm{SL}(2,\mathbb{C})^4 = \bigsqcup_{i=0}^4 Y_i$, we obtain that:

$$\begin{aligned}
e(Y_4) &= e(\mathrm{SL}(2,\mathbb{C})^4) - e(Y_0) - e(Y_1) - e(Y_2) - e(Y_3) \\
&= q^{12} - 2q^{11} - 4q^{10} + 6q^9 - 6q^8 + 18q^7 - 6q^6 - 18q^5 + 15q^4 - 6q^3 + 2q.
\end{aligned}$$

But the E-polynomial of Y_4 can again be obtained using the Hodge monodromy representation $R(\overline{Y}_4/\mathbb{Z}_2)$, using Proposition 2.4.3

$$\begin{aligned}
e(Y_4) &= (q^2 - q)e(\overline{Y}_4/\mathbb{Z}_2) + q\, e(\overline{Y}_4) \\
&= (q^2 - q)((q-2)a - (b + c + d)) + q((q-3)(a + d) - 2(b + c)) \\
&= (q^3 - 2q^2 - q)a - (q^2 + q)(b + c) - 2qd.
\end{aligned} \tag{4.14}$$

Finally, two more equations arise from the E-polynomial of the fiber of $\overline{Y}_4/\mathbb{Z}_2 \longrightarrow \mathbb{C} - \{\pm 2\}$,

$$e(Y_{4,\lambda}) = a + b + c + d, \tag{4.15}$$

and from the Hodge monodromy representation $R(\overline{Y}_4) = (q^9 - 3q^7 + 6q^5 - 6q^4 + 3q^2 - 1)T + (15q^6 - 45q^5 + 45q^4 - 15q^3)N$ given in Proposition 4.3.1. Since $R(\overline{Y}_4) = (a+d)T + (b+c)N$, we get the equation

$$a + d = q^9 - 3q^7 + 6q^5 - 6q^4 + 3q^2 - 1. \tag{4.16}$$

From equations (4.13), (4.14), (4.15) and (4.16), we find

$$\begin{aligned} a &= q^9 - 3q^7 + 6q^5 \\ b &= -45q^5 - 15q^3 \\ c &= 15q^6 + 45q^4 \\ d &= -6q^4 + 3q^2 - 1. \end{aligned}$$

We have proved:

Proposition 4.3.3. $R(\overline{Y}_4/\mathbb{Z}_2) = (q^9 - 3q^7 + 6q^5)T - (45q^5 + 15q^3)S_2 + (15q^6 + 45q^4)S_{-2} + (-6q^4 + 3q^2 - 1)S_0$.

4.4 E-polynomial of the character variety of genus 3

Let $\mathcal{M} = \mathcal{M}_{\mathrm{Id}}(\mathrm{SL}(2,\mathbb{C}))$ be the character variety of a genus 3 complex curve X, i.e, the moduli space of semisimple representations of its fundamental group into $SL(2,\mathbb{C})$. It can be defined as the space

$$\mathcal{M}_{\mathrm{Id}}^{g=3} = V // \mathrm{PGL}(2,\mathbb{C}),$$

where

$$V = \{(A_1, B_1, A_2, B_2, A_3, B_3) \in SL(2,\mathbb{C})^6 \mid [A_1, B_1][A_2, B_2][A_3, B_3] = \mathrm{Id}\}.$$

We stratify V as follows:

- $V_0 = \{(A_1, B_1, A_2, B_2, A_3, B_3) \mid [A_1, B_1][A_2, B_2] = [B_3, A_3] = \mathrm{Id}\}$. Then

$$\begin{aligned} e(V_0) =& e(Y_0)e(X_0) = q^{13} + 5q^{12} + 15q^{11} + 45q^{10} \\ & - 8q^9 - 53q^8 - 32q^7 - 45q^6 + 23q^5 + 44q^4 + q^3 + 4q^2. \end{aligned}$$

- $V_1 = \{(A_1, B_1, A_2, B_2, A_3, B_3) \mid [A_1, B_1][A_2, B_2] = [B_3, A_3] = -\mathrm{Id}\}$. Then

$$e(V_1) = e(Y_1)e(X_1) = q^{12} - 4q^{10} - 30q^9 + 3q^8 + 60q^7 + 3q^6 - 30q^5 - 4q^4 + q^2.$$

- $V_2 = \{(A_1, B_1, A_2, B_2, A_3, B_3) \mid [A_1, B_1][A_2, B_2] = [B_3, A_3] \sim J_+\}$.

$$\begin{aligned} e(V_2) =& e(\mathrm{PGL}(2,\mathbb{C})/U)e(\overline{Y}_2)e(\overline{X}_2) = q^{14} - 2q^{13} - 7q^{12} + 4q^{11} - 16q^{10} \\ & + 84q^9 + 132q^8 - 44q^7 - 65q^6 - 42q^5 - 45q^4. \end{aligned}$$

- $V_3 = \{(A_1, B_1, A_2, B_2, A_3, B_3) \mid [A_1, B_1][A_2, B_2] = [B_3, A_3] \sim J_-\}$.

$$\begin{aligned} e(V_3) =& e(\mathrm{PGL}(2,\mathbb{C})/U)e(\overline{Y}_3)e(\overline{X}_3) = q^{14} + 3q^{13} - 4q^{12} + 3q^{11} + 54q^{10} \\ & + 57q^9 + 84q^8 - 63q^7 - 135q^6. \end{aligned}$$

- $V_4 = \{(A_1, B_1, A_2, B_2, A_3, B_3) \mid [A_1, B_1][A_2, B_2] = [B_3, A_3] \sim \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \lambda \neq 0, \pm 1\}$.

For computing $e(Y_4)$, we define

$$\overline{V}_{4,\lambda} := \left\{ (A_1, B_1, A_2, B_2, A_3, B_3) \mid [A_1, B_1][A_2, B_2] = [B_3, A_3] = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \right\},$$

for $\lambda \neq 0, \pm 1$. There is a fibration $\overline{V}_4 \longrightarrow \mathbb{C} - \{0, \pm 1\}$ with fiber $\overline{V}_{4,\lambda}$. Note that $\overline{V}_{4,\lambda} \cong \overline{Y}_{4,\lambda} \times \overline{X}_{4,\lambda}$, so

$$\begin{aligned} R(\overline{V}_4/\mathbb{Z}_2) =& R(\overline{Y}_4/\mathbb{Z}_2) \otimes R(\overline{X}_4/\mathbb{Z}_2) \\ =& ((q^9 - 3q^7 + 6q^5)T - (45q^5 + 15q^3)S_2 + (15q^6 + 45q^4)S_{-2} \\ & + (-6q^4 + 3q^2 - 1)S_0) \otimes (q^3 T - 3qS_2 + 3q^2 S_{-2} - S_0) \\ =& (q^{12} - 3q^{10} + 51q^8 + 270q^6 + 51q^4 - 3q^2 + 1)T \\ & + (-3q^{10} - 36q^8 - 66q^6 - 36q^4 - 3q^2)S_2 \\ & + (3q^{11} + 6q^9 + 63q^7 + 63q^5 + 6q^3 + 3q)S_{-2} \\ & + (-q^9 - 183q^7 - 183q^5 - q^3)S_0, \end{aligned}$$

which we write as $R(\overline{V}_4/\mathbb{Z}_2) = \tilde{a}T + \tilde{b}S_2 + \tilde{c}S_{-2} + \tilde{d}S_0$. If we apply 2.4.3,

$$\begin{aligned} e(V_4) &= q(q^2 - 2q - 1)\tilde{a} - q(q+1)(\tilde{b} + \tilde{c}) - 2q\tilde{d} \\ &= q^{15} - 2q^{14} - 7q^{13} + 6q^{12} + 51q^{11} - 70q^{10} + 192q^9 \\ &\quad - 171q^8 - 216q^7 + 237q^6 - 24q^5 + 5q^4 + 4q^3 - 5q^2 - q. \end{aligned}$$

From this

$$\begin{aligned} e(V) &= e(V_0) + e(V_1) + e(V_2) + e(V_3) + e(V_4) \\ &= q^{15} - 5q^{13} + q^{12} + 73q^{11} + 9q^{10} + 295q^9 - 5q^8 - 295q^7 - 5q^6 - 73q^5 + 5q^3 - q. \end{aligned} \tag{4.17}$$

4.4.1 Contribution from reducibles

To compute the E-polynomial of $\mathcal{M} = \mathcal{M}(\mathrm{SL}(2,\mathbb{C}))$, we need to take a GIT quotient and differentiate between reducible and irreducible orbits.

In [53, Section 8], this analysis is done in the case of $g = 2$, by stratifying the set of irreducible orbits, and computing the E-polynomial of each stratum. The number of strata increases rapidly with the genus. Therefore, for $g = 3$ we are going to follow the method in [52] which consists on computing the E-polynomial of the reducible locus (which has fewer strata) and substracting it from the total.

A reducible representation given by $(A_1, B_1, A_2, B_2, A_3, B_3)$ is S-equivalent to

$$\left(\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix}, \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2^{-1} \end{pmatrix}, \begin{pmatrix} \lambda_3 & 0 \\ 0 & \lambda_3^{-1} \end{pmatrix}, \begin{pmatrix} \lambda_4 & 0 \\ 0 & \lambda_4^{-1} \end{pmatrix}, \begin{pmatrix} \lambda_5 & 0 \\ 0 & \lambda_5^{-1} \end{pmatrix}, \begin{pmatrix} \lambda_6 & 0 \\ 0 & \lambda_6^{-1} \end{pmatrix} \right), \tag{4.18}$$

under the equivalence relation $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) \sim (\lambda_1^{-1}, \lambda_2^{-1}, \lambda_3^{-1}, \lambda_4^{-1}, \lambda_5^{-1}, \lambda_6^{-1})$ given by the permutation of the eigenvectors. Under the action $\lambda \mapsto \lambda^{-1}$ we have that $e(\mathbb{C}^*)^+ = q$, $e(\mathbb{C}^*)^- = -1$, so we obtain

$$\begin{aligned} e(\mathcal{M}^{red}) &= e((\mathbb{C}^*)^6/\mathbb{Z}_2) \\ &= (e(\mathbb{C}^*)^+)^6 + \binom{6}{2}(e(\mathbb{C}^*)^+)^4(e(\mathbb{C}^*)^-)^2 + \binom{6}{4}(e(\mathbb{C}^*)^+)^2(e(\mathbb{C}^*)^-)^4 + (e(\mathbb{C}^*)^-)^6 \\ &= q^6 + 15q^4 + 15q^2 + 1. \end{aligned}$$

A reducible representation happens when there is a common eigenvector. So in a suitable basis, it is

$$\left(\begin{pmatrix} \lambda_1 & a_1 \\ 0 & \lambda_1^{-1} \end{pmatrix}, \begin{pmatrix} \lambda_2 & a_2 \\ 0 & \lambda_2^{-1} \end{pmatrix}, \begin{pmatrix} \lambda_3 & a_3 \\ 0 & \lambda_3^{-1} \end{pmatrix}, \begin{pmatrix} \lambda_4 & a_4 \\ 0 & \lambda_4^{-1} \end{pmatrix}, \begin{pmatrix} \lambda_5 & a_5 \\ 0 & \lambda_5^{-1} \end{pmatrix}, \begin{pmatrix} \lambda_6 & a_6 \\ 0 & \lambda_6^{-1} \end{pmatrix} \right). \tag{4.19}$$

This is parametrized by $(\mathbb{C}^* \times \mathbb{C})^6$. The condition $[A_1, B_1][A_2, B_2][A_3, B_3] = \mathrm{Id}$ is rewritten as

$$\lambda_2(\lambda_1^2 - 1)a_2 - \lambda_1(\lambda_2^2 - 1)a_1 + \lambda_4(\lambda_3^2 - 1)a_4 - \lambda_3(\lambda_4^2 - 1)a_3 + \lambda_6(\lambda_5^2 - 1)a_6 - \lambda_5(\lambda_6^2 - 1)a_5 = 0. \tag{4.20}$$

There are four cases:

- R_1 given by $(a_1, a_2, a_3, a_4, a_5, a_6) \in \langle(\lambda_1 - \lambda_1^{-1}, \lambda_2 - \lambda_2^{-1}, \lambda_3 - \lambda_3^{-1}, \lambda_4 - \lambda_4^{-1}, \lambda_5 - \lambda_5^{-1}, \lambda_6 - \lambda_6^{-1})\rangle$ and $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) \neq (\pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1)$. Then we can conjugate the representation (4.19) to the diagonal form (4.18). In this case we can suppose all $a_i = 0$, and the stabilizer are the diagonal matrices $D \subset \mathrm{PGL}(2, \mathbb{C})$. There is an action of $\mathbb{Z}_2$ given by interchanging of the two basis vectors, and if we write $A := (\mathbb{C}^*)^6 - \{(\pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1)\}$, the stratum is $(A \times \mathrm{PGL}(2, \mathbb{C})/D)/\mathbb{Z}_2$. Note that $e(A)^+ = q^6 + 15q^4 + 15q^2 - 63$, $e(A)^- = e(A) - e(A)^+ = -(6q^5 + 20q^3 + 6q)$, and $e(\mathrm{PGL}(2, \mathbb{C})/D)^+ = q^2$, $e(\mathrm{PGL}(2, \mathbb{C})/D)^- = q$. So

$$e(R_1) = q^8 + 9q^6 - 5q^4 - 69q^2.$$

- R_2 given by $(a_1, a_2, a_3, a_4, a_5, a_6) \notin \langle(\lambda_1 - \lambda_1^{-1}, \lambda_2 - \lambda_2^{-1}, \lambda_3 - \lambda_3^{-1}, \lambda_4 - \lambda_4^{-1}, \lambda_5 - \lambda_5^{-1}, \lambda_6 - \lambda_6^{-1})\rangle$ and $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) \neq (\pm1, \pm1, \pm1, \pm1, \pm1, \pm1)$. Then (5.12) determines a hyperplane $H \subset \mathbb{C}^6$, and the condition for $(a_1, a_2, a_3, a_4, a_5, a_6)$ defines a line $\ell \subset H$. If $U' \cong D \times U$ denotes the upper triangular matrices, we have a surjective map $A \times (H - \ell) \times \mathrm{PGL}(2, \mathbb{C}) \to R_2$ and the fiber is isomorphic to U'. So

$$\begin{aligned} e(R_2) &= ((q-1)^6 - 64)(q^5 - q)(q^3 - q)/(q^2 - q) \\ &= q^{12} - 5q^{11} + 9q^{10} - 5q^9 - 6q^8 + 14q^7 - 78q^6 - 58q^5 + 5q^4 - 9q^3 + 69q^2 + 63q. \end{aligned}$$

- R_3, given by $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) = (\pm1, \pm1, \pm1, \pm1, \pm1, \pm1)$, $(a_1, a_2, a_3, a_4, a_5, a_6) = (0, 0, 0, 0, 0, 0)$. This is the case where $A_i = B_i = \pm\mathrm{Id}$, $i = 1, 2, 3$, which gives 64 points. Therefore
$$e(R_3) = 64.$$

- R_4, given by $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) = (\pm1, \pm1, \pm1, \pm1, \pm1, \pm1)$, $(a_1, a_2, a_3, a_4, a_5, a_6) \neq (0, 0, 0, 0, 0, 0)$. In this case, there is at least a matrix of Jordan type. The diagonal matrices act projectivizing the set $(a_1, a_2, a_3, a_4, a_5, a_6) \in \mathbb{C} - \{(0, 0, 0, 0, 0, 0)\}$ and the stabilizer is isomorphic to U. So
$$e(R_4) = 64\, e(\mathbb{P}^5) e(\mathrm{PGL}(2, \mathbb{C})/U) = 64q^7 + 64q^6 - 64q - 64.$$

Adding all up, we have

$$\begin{aligned} e(V^{red}) = e(R_1) + e(R_2) + e(R_3) + e(R_4) &= q^{12} - 5q^{11} \\ &+ 9q^{10} - 5q^9 - 5q^8 + 78q^7 - 5q^6 - 58q^5 - 9q^3 - q, \end{aligned}$$

and hence

$$e(V^{irr}) = e(V) - e(V^{red}) = q^{15} - 5q^{13} + 78q^{11} + 300q^9 - 373q^7 - 15q^5 + 14q^3,$$

and thus

$$e(\mathcal{M}_{\mathrm{Id}}^{irr}) = e(V^{irr})/e(\mathrm{PGL}(2, \mathbb{C})) = q^{12} - 4q^{10} + 74q^8 + 374q^6 + q^4 - 14q^2.$$

Finally, we have

$$e(\mathcal{M}_{\mathrm{Id}}^{g=3}) = e(\mathcal{M}_{\mathrm{Id}}^{red}) + e(\mathcal{M}_{\mathrm{Id}}^{irr}) = q^{12} - 4q^{10} + 74q^8 + 375q^6 + 16q^4 + q^2 + 1.$$

Chapter 5

$SL(2,\mathbb{C})$-character varieties of surfaces of genus $g \geq 3$

5.1 Introduction

The main theorem of this chapter is the computation of the E-polynomials of the $SL(2,\mathbb{C})$-character varieties of surface groups and arbitrary genus. It can be considered the central result of this dissertation and relies on the tools developed in Chapter 2 and the results of Chapters 3 and 4.

Theorem 5.1.1. *Let X be a complex curve of genus $g \geq 1$. Let $\mathcal{M}_C^g = \mathcal{M}_C^g(\mathrm{SL}(2,\mathbb{C}))$ be the character variety corresponding to $C \in \mathrm{SL}(2,\mathbb{C})$. The E-polynomials of $\mathcal{M}_C^g$ are:*

$$
\begin{aligned}
e(\mathcal{M}_{\mathrm{Id}}^g) &= (q^3-q)^{2g-2} + (q^2-1)^{2g-2} - q(q^2-q)^{2g-2} - 2^{2g}q^{2g-2} \\
&\quad + \frac{1}{2}q^{2g-2}(q+2^{2g}-1)((q+1)^{2g-2}+(q-1)^{2g-2}) + \frac{1}{2}q((q+1)^{2g-1}+(q-1)^{2g-1}) \\
e(\mathcal{M}_{-\mathrm{Id}}^g) &= (q^3-q)^{2g-2} + (q^2-1)^{2g-2} - 2^{2g-1}(q^2+q)^{2g-2} + (2^{2g-1}-1)(q^2-q)^{2g-2} \\
e(\mathcal{M}_{J_+}^g) &= (q^3-q)^{2g-2}(q^2-1) + (2^{2g-1}-1)(q-1)(q^2-q)^{2g-2} - 2^{2g-1}(q+1)(q^2+q)^{2g-2} \\
&\quad + \frac{1}{2}q^{2g-2}(q-1)\left((q-1)^{2g-1}-(q+1)^{2g-1}\right) \\
e(\mathcal{M}_{J_-}^g) &= (q^3-q)^{2g-2}(q^2-1) + (2^{2g-1}-1)(q-1)(q^2-q)^{2g-2} + 2^{2g-1}(q+1)(q^2+q)^{2g-2} \\
e(\mathcal{M}_{\xi_\lambda}^g) &= (q^3-q)^{2g-2}(q^2+q) + (q^2-1)^{2g-2}(q+1) + (2^{2g}-2)(q^2-q)^{2g-2}q,
\end{aligned}
$$

for $J_+ = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $J_- = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$ and $\xi_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, $\lambda \neq 0, \pm 1$, and with $q = uv$.

This theorem generalizes the formulas of [53] for $g = 1, 2$ given in Chapter 3 and generalizes the formulas given in Chapter 4 for $e(\mathcal{M}_{\mathrm{Id}}^{g=3})$ and $e(\mathcal{M}_{-\mathrm{Id}}^{g=3})$. The formula for $e(\mathcal{M}_{-\mathrm{Id}})$ and any g coincides with that of [60].

The basic idea is to use the E-polynomials of Chapters 3 and 4 as building blocks, and to decompose X^g as a connected sum in different ways ($X^g = X^k \# X^h, k + h = g$), each of which gives a stratification of the representation space X^g. From the information for X^g one gets information for X^{g-1} with a hole, and this is used in turn to compute the E-polynomial corresponding to $X^{g+1} = X^{g-1}\#X^2$. The E-polynomials of X^1, X^2 come into play here. The general formulas are obtained inductively.

The starting point for induction is the representation space X^3 of a complex curve of genus $g = 3$ (with no puncture), which was treated in Chapter 4 and has its special features. In that case, the techniques to compute E-polynomials of analytically locally trivial fibrations use a base of dimension 2, but in all other cases ($g = 1, 2$ in Chapter 3, and the induction for $g \geq 4$ treated here) a base of dimension 1 suffices.

Some consequences can be obtained from the proof and the formulas of Theorem 5.1.1. The first one is

Theorem 5.1.2. *All character varieties $\mathcal{M}_C(SL(2,\mathbb{C}))$ are of balanced type.*

The formulas in Theorem 5.1.1 allow us to prove the following relation of the E-polynomials of various character varieties, conjectured by T. Hausel.

Corollary 5.1.3. *For any genus $g \geq 1$, we have*

$$e(\mathcal{M}_{J_-}) + (q+1)e(\mathcal{M}_{-\mathrm{Id}}) = e(\mathcal{M}_{\xi_\lambda}).$$

In this chapter, the behaviour of the E-polynomial of the parabolic character variety ($G = \mathrm{SL}(2,\mathbb{C})$)

$$\mathcal{M}_{\xi_\lambda} = \mathcal{M}_{\xi_\lambda}(G) = \{(A_1, B_1, \ldots, A_g, B_g) \in G^{2g} \mid \prod_{i=1}^{g}[A_i, B_i] = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}\}//G,$$

is also described when λ varies in $\mathbb{C} - \{0, \pm 1\}$. Specifically,

Theorem 5.1.4. *Let X be a curve of genus $g \geq 1$. Then*

$$\begin{aligned} R(\mathcal{M}_{\xi_\lambda}) = &\left((q^3-q)^{2g-2}(q^2+q) + (q+1)(q^2-1)^{2g-2} - q(q^2-q)^{2g-2}\right) T \\ &+ \left((2^{2g}-1)q(q^2-q)^{2g-2}\right) N, \end{aligned}$$

As in previous chapters, the E-polynomial of the invariant part of the cohomology is the polynomial accompanying T, and the E-polynomial of the non-invariant part is the polynomial accompanying N, where T, N are the trivial and non trivial representations respectively.

Finally, we end up giving some topological consequences of Theorem 5.1.1 in Section 5.9. The present arguments can be used to compute the E-polynomials of the $\mathrm{PGL}(2,\mathbb{C})$-character varieties of surface groups for arbitrary genus, which will appear in [55].

5.2 Stratifying the space of representations

Let $g \geq 1$ be any natural number. We define the following sets:

- $\overline{X}_0^g = \{(A_1, B_1, \ldots, A_g, B_g) \in \mathrm{SL}(2,\mathbb{C})^{2g} \mid \prod_{i=1}^g [A_i, B_i] = \mathrm{Id}\}$.
- $\overline{X}_1^g = \{(A_1, B_1, \ldots, A_g, B_g) \in \mathrm{SL}(2,\mathbb{C})^{2g} \mid \prod_{i=1}^g [A_i, B_i] = -\mathrm{Id}\}$.
- $\overline{X}_2^g = \{(A_1, B_1, \ldots, A_g, B_g) \in \mathrm{SL}(2,\mathbb{C})^{2g} \mid \prod_{i=1}^g [A_i, B_i] = J_+ = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\}$.
- $\overline{X}_3^g = \{(A_1, B_1, \ldots, A_g, B_g) \in \mathrm{SL}(2,\mathbb{C})^{2g} \mid \prod_{i=1}^g [A_i, B_i] = J_- = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}\}$.
- $\overline{X}_{4,\lambda}^g = \{(A_1, B_1, \ldots, A_g, B_g) \in \mathrm{SL}(2,\mathbb{C})^{2g} \mid \prod_{i=1}^g [A_i, B_i] = \xi_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}\}$, where $\lambda \in \mathbb{C} - \{0, \pm 1\}$.
- $\overline{X}_4^g = \{(A_1, B_1, \ldots, A_g, B_g, \lambda) \in \mathrm{SL}(2,\mathbb{C})^{2g} \times (\mathbb{C} - \{0, \pm 1\}) \mid \prod_{i=1}^g [A_i, B_i] = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}\}$.

There is a natural fibration

$$\overline{X}_4^g \longrightarrow \mathbb{C} - \{0, \pm 1\}$$

whose fibers are $\overline{X}_{4,\lambda}^g$. There is an action of $\mathbb{Z}_2$ on $\overline{X}_4$ given by

$$(A_1, \ldots, B_g, \lambda) \mapsto (P_0^{-1} A_1 P_0, \ldots, P_0^{-1} B_g P_0, \lambda^{-1}),$$

with $P_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and an induced fibration

$$\overline{X}_4^g / \mathbb{Z}_2 \longrightarrow (\mathbb{C} - \{0, \pm 1\})/\mathbb{Z}_2 \cong \mathbb{C} - \{\pm 2\},$$

where the basis is parametrized by $s = \lambda + \lambda^{-1}$.

Recall that for $g = 1, 2$, the monodromy group is $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$, generated by the loops $\gamma_{\pm 2}$ around the punctures ± 2 of $\mathbb{C} - \{\pm 2\}$. The ring $R(\Gamma)$ is generated by the trivial representation T, the representations $S_{\pm 2}$ which are non-trivial around ± 2 and trivial around ∓ 2, and the representation $S_0 = S_2 \otimes S_{-2}$.

Now we shall set up an induction. Assume that for all $k < g$, the Hodge monodromy representation of $\overline{X}_4^k / \mathbb{Z}_2$ is in $R(\Gamma)[q]$. We write

$$R(\overline{X}_4^k / \mathbb{Z}_2) = a_k T + b_k S_2 + c_k S_{-2} + d_k S_0,$$

for some polynomials $a_k, b_k, c_k, d_k \in \mathbb{Z}[q]$.

Take $k, h < g$. Fix some $C = \mathrm{Id}, -\mathrm{Id}, J_+, J_-$ or ξ_λ. Then

$$\prod_{i=1}^{k+h} [A_i, B_i] = C \iff \prod_{i=1}^{k} [A_i, B_i] = C \prod_{i=1}^{h} [B_{k+i}, A_{k+i}]. \tag{5.1}$$

5.3 Computation of $e(\overline{X}_0^{k+h})$

Using (5.1), we stratify $\overline{X}_0^{k+h} = \bigsqcup W_i$, where

- $W_0 = \{(A_1, B_1, \ldots, A_{k+h}, B_{k+h}) \mid \prod_{i=1}^k [A_i, B_i] = \prod_{i=1}^h [B_{k+i}, A_{k+i}] = \mathrm{Id}\} \cong \overline{X}_0^k \times \overline{X}_0^h$.
- $W_1 = \{(A_1, B_1, \ldots, A_{k+h}, B_{k+h}) \mid \prod_{i=1}^k [A_i, B_i] = \prod_{i=1}^h [B_{k+i}, A_{k+i}] = -\mathrm{Id}\} \cong \overline{X}_1^k \times \overline{X}_1^h$.
- $W_2 = \{(A_1, B_1, \ldots, A_{k+h}, B_{k+h}) \mid \prod_{i=1}^k [A_i, B_i] = \prod_{i=1}^h [B_{k+i}, A_{k+i}] \sim J_+\}$, isomorphic to $\mathrm{PGL}(2,\mathbb{C})/U \times \overline{X}_2^k \times \overline{X}_2^h$, where $U = \{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\} \cong \mathbb{C}$.
- $W_3 = \{(A_1, B_1, \ldots, A_{k+h}, B_{k+h}) \mid \prod_{i=1}^k [A_i, B_i] = \prod_{i=1}^h [B_{k+i}, A_{k+i}] \sim J_-\}$, isomorphic to $\mathrm{PGL}(2,\mathbb{C})/U \times \overline{X}_3^k \times \overline{X}_3^h$.
- $W_4 = \{(A_1, B_1, \ldots, A_{k+h}, B_{k+h}) \mid \prod_{i=1}^k [A_i, B_i] = \prod_{i=1}^h [B_{k+i}, A_{k+i}] \sim \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \lambda \neq 0, \pm 1\}$.

To compute $e(W_4)$, we define

$$\overline{W}_4 = \{(A_1, B_1, \ldots, A_{k+h}, B_{k+h}, \lambda) \mid \prod_{i=1}^k [A_i, B_i] = \prod_{i=1}^h [B_{k+i}, A_{k+i}] = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \lambda \neq 0, \pm 1\},$$

which produces the fibration

$$\overline{W}_4/\mathbb{Z}_2 \to \mathbb{C} - \{\pm 2\},$$

whose Hodge monodromy representation is

$$\begin{aligned} R(\overline{W}_4/\mathbb{Z}_2) &= R(\overline{X}_4^k/\mathbb{Z}_2) \otimes R(\overline{X}_4^h/\mathbb{Z}_2) \\ &= (a_k a_h + b_k b_h + c_k c_h + d_k d_h)T + (a_k b_h + b_k a_h + c_k d_h + d_k c_h)S_2 \\ &\quad + (a_k c_h + b_k d_h + c_k a_h + d_k b_h)S_{-2} + (a_k d_h + b_k c_h + c_k b_h + d_k a_h)S_0. \end{aligned} \tag{5.2}$$

From $R(\overline{W}_4/\mathbb{Z}_2)$ we can obtain $e(W_4)$ thanks to Proposition 2.4.3. For brevity, write $R(\overline{W}_4/\mathbb{Z}_2) = AT + BS_2 + CS_{-2} + DS_0$, where

$$\begin{aligned} A &= a_k a_h + b_k b_h + c_k c_h + d_k d_h \\ B &= a_k b_h + b_k a_h + c_k d_h + d_k c_h \\ C &= a_k c_h + b_k d_h + c_k a_h + d_k b_h \\ D &= a_k d_h + b_k c_h + c_k b_h + d_k a_h \end{aligned} \tag{5.3}$$

First, using Corollary 2.3.5, we have that $e(\overline{W}_4/\mathbb{Z}_2) = (q-2)A - (B + C + D)$. On the other hand, the $2:1$-cover $\mathbb{C} - \{0, \pm 1\} \to \mathbb{C} - \{\pm 2\}$ allows us to deduce that $R(\overline{W}_4) =$

$(A+D)T+(B+C)N$, where T is the trivial representation, and N is the representation which is non-trivial and of order two around the origin. So using (2.3.5) again, we have that $e(\overline{W}_4)=(q-3)(A+D)-2(B+C)$.

Now note that

$$W_4 \cong (\mathrm{PGL}(2,\mathbb{C})/D \times \overline{W}_4)/\mathbb{Z}_2,$$

where $D\cong\mathbb{C}^*$ are the diagonal matrices. By Proposition 2.4.2, we have that $e(\mathrm{PGL}(2,\mathbb{C})/D)^+ = q^2$ and $e(\mathrm{PGL}(2,\mathbb{C})/D)^- = q$. Using Proposition 2.3.8,

$$\begin{aligned} e(W_4) &= q^2 e(\overline{W}_4)^+ + q\,e(\overline{W}_4)^- \\ &= q^2 e(\overline{W}_4/\mathbb{Z}_2) + q(e(\overline{W}_4) - e(\overline{W}_4/\mathbb{Z}_2)) \\ &= (q^2-q)e(\overline{W}_4/\mathbb{Z}_2) + q\,e(\overline{W}_4) \\ &= (q^2-q)((q-2)A-(B+C+D)) + q((q-3)(A+D)-2(B+C)) \\ &= (q^3-2q^2-q)A-(q^2+q)(B+C)-2qD. \end{aligned} \tag{5.4}$$

All together, recalling also that $e(\mathrm{PGL}(2,\mathbb{C}))=q^3-q$ and so $e(\mathrm{PGL}(2,\mathbb{C})/U)=q^2-1$, we have

$$\begin{aligned} e(\overline{X}_0^{k+h}) &= e(\overline{X}_0^k)e(\overline{X}_0^h) + e(\overline{X}_1^k)e(\overline{X}_1^h) \\ &\quad + (q^2-1)e(\overline{X}_2^k)e(\overline{X}_2^h) + (q^2-1)e(\overline{X}_3^k)e(\overline{X}_3^h) + e(W_4). \end{aligned} \tag{5.5}$$

Setting $k=g-1$, $h=1$, and substituting the values A,B,C,D from (5.3) into (5.4), and then the values of $e(\overline{X}_j^1)$ and a_1,b_1,c_1,d_1 from Section 5.2, we have

$$\begin{aligned} e_0^g = e(\overline{X}_0^g) = {} & (q^4+4q^3-q^2-4q)e_0^{g-1} + (q^3-q)e_1^{g-1} \\ & + (q^5-2q^4-4q^3+2q^2+3q)e_2^{g-1} + (q^5+3q^4-q^3-3q^2)e_3^{g-1} \\ & + (q^6-2q^5-4q^4+3q^2+2q)a_{g-1} + (-q^5-4q^4+4q^2+q)b_{g-1} \\ & + (2q^5-7q^4-3q^3+7q^2-q)c_{g-1} + (-5q^4-q^3+5q^2-q)d_{g-1} \end{aligned} \tag{α}$$

5.4 Computation of $e(\overline{X}_1^{k+h})$

We do something similar to the previous case. We stratify $\overline{X}_1^{k+h} = \bigsqcup W_i'$, where

- $W_0' = \{(A_1,B_1,\dots,A_{k+h},B_{k+h}) \mid \prod_{i=1}^k [A_i,B_i] = -\prod_{i=1}^h [B_{k+i},A_{k+i}] = \mathrm{Id}\} \cong \overline{X}_0^k \times \overline{X}_1^h$.
- $W_1' = \{(A_1,B_1,\dots,A_{k+h},B_{k+h}) \mid \prod_{i=1}^k [A_i,B_i] = -\prod_{i=1}^h [B_{k+i},A_{k+i}] = -\mathrm{Id}\} \cong \overline{X}_1^k \times \overline{X}_0^h$.

- $W_2' = \{(A_1, B_1, \ldots, A_{k+h}, B_{k+h}) \mid \prod_{i=1}^k [A_i, B_i] = -\prod_{i=1}^h [B_{k+i}, A_{k+i}] \sim J_+\} \cong \mathrm{PGL}(2,\mathbb{C})/U \times \overline{X}_2^k \times \overline{X}_3^h$, where $U = \{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\} \cong \mathbb{C}$.
- $W_3' = \{(A_1, B_1, \ldots, A_{k+h}, B_{k+h}) \mid \prod_{i=1}^k [A_i, B_i] = -\prod_{i=1}^h [B_{k+i}, A_{k+i}] \sim J_-\} \cong \mathrm{PGL}(2,\mathbb{C})/U \times \overline{X}_3^k \times \overline{X}_2^h$.
- $W_4' = \{(A_1, B_1, \ldots, A_{k+h}, B_{k+h}) \mid \prod_{i=1}^k [A_i, B_i] = -\prod_{i=1}^h [B_{k+i}, A_{k+i}] \sim \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}\}$, where $\lambda \neq 0, \pm 1$.

To compute $e(W_4')$, we define $\overline{W}_4' = \{(A_1, B_1, \ldots, A_{k+h}, B_{k+h}, \lambda) \mid \prod_{i=1}^k [A_i, B_i] = -\prod_{i=1}^h [B_{k+i}, A_{k+i}] = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \lambda \neq 0, \pm 1\}$, which produces the fibration

$$\overline{W}_4'/\mathbb{Z}_2 \to \mathbb{C} - \{\pm 2\},$$

whose Hodge monodromy representation is given as (where $\tau(\lambda) = -\lambda$),

$$\begin{aligned} R(\overline{W}_4'/\mathbb{Z}_2) &= R(\overline{X}_4^k/\mathbb{Z}_2) \otimes \tau^* R(\overline{X}_4^h/\mathbb{Z}_2) \\ &= (a_k T + b_k S_2 + c_k S_{-2} + d_k S_0) \otimes (a_h T + c_h S_2 + b_h S_{-2} + d_h S_0) \\ &= (a_k a_h + b_k c_h + c_k b_h + d_k d_h) T + (a_k c_h + b_k a_h + c_k d_h + d_k b_h) S_2 \\ &\quad + (a_k b_h + b_k d_h + c_k a_h + d_k c_h) S_{-2} + (a_k d_h + b_k b_h + c_k c_h + d_k a_h) S_0 \end{aligned} \tag{5.6}$$

We write $R(\overline{W}_4'/\mathbb{Z}_2) = A'T + B'S_2 + C'S_{-2} + D'S_0$, with

$$\begin{aligned} A' &= a_k a_h + b_k c_h + c_k b_h + d_k d_h \\ B' &= a_k c_h + b_k a_h + c_k d_h + d_k b_h \\ C' &= a_k b_h + b_k d_h + c_k a_h + d_k c_h \\ D' &= a_k d_h + b_k b_h + c_k c_h + d_k a_h \end{aligned} \tag{5.7}$$

We use (5.4) to get

$$e(W_4') = (q^3 - 2q^2 - q)A' - (q^2 + q)(B' + C') - 2qD'.$$

Finally

$$\begin{aligned} e(\overline{X}_1^{k+h}) &= e(\overline{X}_0^k) e(\overline{X}_1^h) + e(\overline{X}_1^k) e(\overline{X}_0^h) \\ &\quad + (q^2 - 1) e(\overline{X}_2^k) e(\overline{X}_3^h) + (q^2 - 1) e(\overline{X}_3^k) e(\overline{X}_2^h) + e(W_4'). \end{aligned} \tag{5.8}$$

Setting $k = g-1$, $h = 1$, and substituting the values A', B', C', D' from (5.7) and the values of $e(\overline{X}_j^1)$ and a_1, b_1, c_1, d_1 from Section 5.2, we have

$$\begin{aligned} e_1^g = e(\overline{X}_1^g) = &(q^3-q)e_0^{g-1} + (q^4+4q^3-q^2-4q)e_1^{g-1} \\ &+ (q^5+3q^4-q^3-3q^2)e_2^{g-1} + (q^5-2q^4-4q^3+2q^2+3q)e_3^{g-1} \\ &+ (q^6-2q^5-4q^4+3q^2+2q)a_{g-1} + (2q^5-7q^4-3q^3+7q^2+q)b_{g-1} \\ &+ (-q^5-4q^4+4q^2+q)c_{g-1} + (-5q^4-q^3+5q^2+q)d_{g-1}. \end{aligned} \tag{β}$$

5.5 Computation of $e(\overline{X}_2^{k+h})$

Now we consider

$$Z = \{(A_1, B_1, \ldots, A_{k+h}, B_{k+h}) \mid \prod_{i=1}^{k}[A_i, B_i] = J_+ \prod_{i=1}^{h}[B_{k+i}, A_{k+i}]\}.$$

We write

$$\nu = \prod_{i=1}^{h}[B_{k+i}, A_{k+i}] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \delta = \prod_{i=1}^{k}[A_i, B_i] = J_+\nu = \begin{pmatrix} a+c & b+d \\ c & d \end{pmatrix}.$$

Let $t_1 = \operatorname{tr}\nu$, $t_2 = \operatorname{tr}\delta$. Note that $c = t_2 - t_1$.

We use the stratification defined as in [53, Section 11], according to the values of the pair (t_1, t_2).

- Z_1 given by $(t_1, t_2) = (2, 2)$. In this case $c = 0$, $a = d = 1, b \in \mathbb{C}$. If $b \neq 0, -1$, both are of Jordan type, whereas if $b = 0, 1$ one of them is of Jordan type and the other is equal to Id. We get

$$e(Z_1) = (q-2)e(\overline{X}_2^k)e(\overline{X}_2^h) + e(\overline{X}_2^k)e(\overline{X}_0^h) + e(\overline{X}_0^k)e(\overline{X}_2^h).$$

- Z_2 given by $(t_1, t_2) = (-2, -2)$. Analogously

$$e(Z_2) = (q-2)e(\overline{X}_3^k)e(\overline{X}_3^h) + e(\overline{X}_3^k)e(\overline{X}_1^h) + e(\overline{X}_1^k)e(\overline{X}_3^h).$$

- Z_3 given by $(t_1, t_2) = (2, -2), (-2, 2)$. Now $c \neq 0$. The action of $U \cong \mathbb{C}$ allows to fix $d = 0$. Both ν, δ are of Jordan type. So we obtain

$$e(Z_3) = q(e(\overline{X}_2^k)e(\overline{X}_3^h) + e(\overline{X}_3^k)e(\overline{X}_2^h)).$$

- Z_4 given by the subcases:

– $Z_{4,1}$ given by $(2,t_2), t_2 \neq \pm 2$ and $(-2,t_2), t_2 \neq \pm 2$. We focus on the first case. It must be $c \neq 0$. The group $U \cong \mathbb{C}$ acts freely on the matrix ν, which is of Jordan type. Note also that δ is diagonalizable. The second case is similar with $\nu \sim J_-$. We thus get

$$e(Z_{4,1}) = q(e(\overline{X}_2^h) + e(\overline{X}_3^h))e(\overline{X}_4^k/\mathbb{Z}_2).$$

– $Z_{4,2}$ given by $(t_1,2), t_1 \neq \pm 2$ and $(t_1,-2), t_1 \neq \pm 2$. It is completely similar, interchanging the roles of ν and δ.

$$e(Z_{4,2}) = q(e(\overline{X}_2^k) + e(\overline{X}_3^k))e(\overline{X}_4^h/\mathbb{Z}_2).$$

- Z_5 corresponding to $t_1 = t_2 \neq \pm 2$. Now

$$\eta = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, \ \delta = \begin{pmatrix} a & b + a^{-1} \\ 0 & a^{-1} \end{pmatrix},$$

Therefore $e(Z_5) = q\, e(\overline{Z}_5)$, where $\overline{Z}_5$ is a fibration over $a \in \mathbb{C} - \{0, \pm 1\}$ whose fibers are $\overline{X}_{4,a}^k \times \overline{X}_{4,a}^h$. Thus the Hodge monodromy representation is given in (5.2),

$$\begin{aligned} R(\overline{Z}_5/\mathbb{Z}_2) &= AT + BS_2 + CS_{-2} + DS_0, \\ R(\overline{Z}_5) &= (A+D)T + (B+C)N, \\ e(Z_5) &= q\, e(\overline{Z}_5) = q((q-3)(A+D) - 2(B+C)). \end{aligned}$$

- Z_6 corresponding to the open stratum $t_1, t_2 \neq \pm 2, t_1 \neq t_2$. As $c \neq 0$, we can arrange $d = 0$ by using the action of $U \cong \mathbb{C}$. Both δ and ν are diagonalizable matrices. If we ignore for a while the condition $t_1 \neq t_2$, the total space is isomorphic to $\mathbb{C} \times \overline{X}_4^k/\mathbb{Z}_2 \times \overline{X}_4^h/\mathbb{Z}_2$. The fibration over the diagonal (t_1,t_1) has total space isomorphic $\mathbb{C} \times (\overline{Z}_5/\mathbb{Z}_2)$. Thus

$$\begin{aligned} e(\overline{Z}_5/\mathbb{Z}_2) &= (q-2)A - (B+C+D), \\ e(Z_6) &= q(e(\overline{X}_4^k/\mathbb{Z}_2)e(\overline{X}_4^h/\mathbb{Z}_2) - e(\overline{Z}_5/\mathbb{Z}_2)). \end{aligned}$$

Adding all up,

$$\begin{aligned} e(\overline{X}_2^{k+h}) =& e(\overline{X}_2^k)e(\overline{X}_0^h) + e(\overline{X}_0^k)e(\overline{X}_2^h) - 2e(\overline{X}_2^k)e(\overline{X}_2^h) + e(\overline{X}_3^k)e(\overline{X}_1^h) + e(\overline{X}_1^k)e(\overline{X}_3^h) \\ & - 2e(\overline{X}_3^k)e(\overline{X}_3^h) + q\,(e(\overline{X}_2^k) + e(\overline{X}_3^k) + e(\overline{X}_4^k/\mathbb{Z}_2))(e(\overline{X}_2^h) + e(\overline{X}_3^h) + e(\overline{X}_4^h/\mathbb{Z}_2)) \\ & + q\,(e(\overline{Z}_5) - e(\overline{Z}_5/\mathbb{Z}_2)). \end{aligned}$$

Setting $k = g-1$, $h = 1$, and substituting the values A, B, C, D from (5.3) and the values of $e(\overline{X}_j^1)$ and a_1, b_1, c_1, d_1 from Section 5.2, we have

$$\begin{aligned} e_2^g = e(\overline{X}_2^g) = {} & (q^3 - 2q^2 - 3q)e_0^{g-1} + (q^3 + 3q^2)e_1^{g-1} \\ & + (q^5 + q^4 + 3q^2 + 3q)e_2^{g-1} + (q^5 - 3q^3 - 6q^2)e_3^{g-1} \\ & + (q^6 - 2q^5 - 3q^4 + q^3 + 3q^2)a_{g-1} + (-q^5 + 2q^4 - 4q^3 + 3q^2)b_{g-1} \\ & + (-q^5 - q^4 - 4q^3 + 6q^2)c_{g-1} + (-2q^4 - q^3 + 3q^2)d_{g-1}. \end{aligned} \tag{γ}$$

5.6 Computation of $e(\overline{X}_3^{k+h})$

This is similar to the previous case. Consider

$$Z' = \{(A_1, B_1, \dots, A_{k+h}, B_{k+h}) \mid \prod_{i=1}^{k}[A_i, B_i] = J_- \prod_{i=1}^{h}[B_{k+i}, A_{k+i}]\}.$$

We write

$$\nu = \prod_{i=1}^{h}[B_{k+i}, A_{k+i}] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \delta = \prod_{i=1}^{k}[A_i, B_i] = J_-\nu = \begin{pmatrix} -a+c & -b+d \\ -c & -d \end{pmatrix}.$$

Let $t_1 = \operatorname{tr}\nu$, $t_2 = \operatorname{tr}\delta$. In this case, $c = t_2 + t_1$. Stratifying as in Section 5.5, we get

- Z_1' given by $(t_1, t_2) = (2, -2)$. We obtain

$$e(Z_1') = (q-2)e(\overline{X}_2^h)e(\overline{X}_3^k) + e(\overline{X}_2^h)e(\overline{X}_1^k) + e(\overline{X}_0^h)e(\overline{X}_3^k).$$

- Z_2' given by $(t_1, t_2) = (-2, 2)$. Analogously

$$e(Z_2') = (q-2)e(\overline{X}_3^h)e(\overline{X}_2^k) + e(\overline{X}_3^h)e(\overline{X}_0^k) + e(\overline{X}_1^h)e(\overline{X}_2^k).$$

- Z_3' given by the two values $(t_1, t_2) = (2, 2), (-2, -2)$. We get

$$e(Z_3') = q(e(\overline{X}_2^k)e(\overline{X}_2^h) + e(\overline{X}_3^k)e(\overline{X}_3^h)).$$

- Z_4', divided into two possible cases:
  - $Z_{4,1}'$ given by the lines $t_1 = 2$, $t_2 \neq \pm 2$ and $t_1 = -2$, $t_2 \neq \pm 2$. We get

$$e(Z_{4,1}') = q(e(\overline{X}_2^h) + e(\overline{X}_3^h))e(\overline{X}_4^k/\mathbb{Z}_2).$$

  - $Z_{4,2}'$ given by the lines $t_2 = 2$, $t_1 \neq \pm 2$ and $t_2 = -2$, $t_1 \neq \pm 2$. We obtain

$$e(Z_{4,2}') = q(e(\overline{X}_2^k) + e(\overline{X}_3^k))e(\overline{X}_4^h/\mathbb{Z}_2).$$

- Z'_5 corresponding to $t_1 = -t_2 \neq \pm 2$. So $c = 0$ and

$$\eta = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, \ \delta = \begin{pmatrix} -a & -b + a^{-1} \\ 0 & -a^{-1} \end{pmatrix}.$$

  Therefore $e(Z'_5) = q\, e(\overline{Z}'_5)$, where $\overline{Z}'_5$ is a fibration over $a \in \mathbb{C} - \{0, \pm 1\}$ whose fibers are $\overline{X}^k_{4,a} \times \overline{X}^h_{4,-a}$. Thus the Hodge monodromy representation is given in (5.6),

$$\begin{aligned} R(\overline{Z}'_5/\mathbb{Z}_2) &= A'T + B'S_2 + C'S_{-2} + D'S_0, \\ R(\overline{Z}'_5) &= (A' + D')T + (B' + C')N, \\ e(Z'_5) &= q\, e(\overline{Z}'_5) = q((q-3)(A' + D') - 2(B' + C')). \end{aligned}$$

- Z'_6 corresponding to the open stratum $t_1, t_2 \neq \pm 2$, $t_1 \neq -t_2$. The action of $U \cong \mathbb{C}$ can be used to set $d = 0$. The total space, ignoring the condition $t_1 \neq -t_2$, gives a contribution of $e(\overline{X}^k_4)e(\overline{X}^h_4)$. The fibration over the diagonal $(t_1, -t_1)$ has total space isomorphic $\mathbb{C} \times (\overline{Z}'_5/\mathbb{Z}_2)$. Thus

$$\begin{aligned} e(\overline{Z}'_5/\mathbb{Z}_2) &= (q-2)A' - (B' + C' + D'), \\ e(Z'_6) &= q(e(\overline{X}^k_4/\mathbb{Z}_2)e(\overline{X}^h_4/\mathbb{Z}_2) - e(\overline{Z}'_5/\mathbb{Z}_2)). \end{aligned}$$

Adding all up,

$$\begin{aligned} e(\overline{X}^{k+h}_3) = {} & e(\overline{X}^k_2)e(\overline{X}^h_1) + e(\overline{X}^k_0)e(\overline{X}^h_3) - 2e(\overline{X}^k_2)e(\overline{X}^h_2) + e(\overline{X}^k_3)e(\overline{X}^h_0) + e(\overline{X}^k_1)e(\overline{X}^h_2) \\ & - 2e(\overline{X}^k_2)e(\overline{X}^h_3) + q\,(e(\overline{X}^k_2) + e(\overline{X}^k_3) + e(\overline{X}^k_4/\mathbb{Z}_2))(e(\overline{X}^h_2) + e(\overline{X}^h_3) + e(\overline{X}^h_4/\mathbb{Z}_2)) \\ & + q\,(e(\overline{Z}'_5) - e(\overline{Z}'_5/\mathbb{Z}_2)). \end{aligned}$$

Setting $k = g-1$, $h = 1$, and substituting the values A', B', C', D' from (5.7) and the values of $e(\overline{X}^1_j)$ and a_1, b_1, c_1, d_1 from Section 5.2, we have:

$$\begin{aligned} e^g_3 = e(\overline{X}^g_3) = {} & (q^3 + 3q^2)e^{g-1}_0 + (q^3 - 2q^2 - 3q)e^{g-1}_1 \\ & + (q^5 - 3q^3 - 6q^2)e^{g-1}_2 + (q^5 + q^4 + 3q^2 + 3q)e^{g-1}_3 \\ & + (q^6 - 2q^5 - 3q^4 + q^3 + 3q^2)a_{g-1} + (-q^5 - q^4 - 4q^3 + 6q^2)b_{g-1} \\ & + (-q^5 + 2q^4 - 4q^3 + 3q^2)c_{g-1} + (-2q^4 - q^3 + 3q^2)d_{g-1} \end{aligned} \tag{η}$$

5.7 Computation of $R(\overline{X}^{k+h}_4)$

Now we move to the stratum $\overline{X}^g_4$. This one is controlled by a Hodge monodromy representation $R(\overline{X}^g_4/\mathbb{Z}_2)$. We start by computing $R(\overline{X}^g_4)$. As before, we write

$$\overline{X}^{k+h}_4 = \{(A_1, B_1, \ldots, A_{k+h}, B_{k+h}, \lambda) \mid \prod_{i=1}^{k}[A_i, B_i] = \xi_\lambda \prod_{i=1}^{h}[B_{k+i}, A_{k+i}], \lambda \neq 0, \pm 1\},$$

where $\xi_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$. We are going to study the fibration $\overline{X}_4^{k+h} \to \mathbb{C} - \{0, \pm 1\}$, with fiber $\overline{X}_{4,\lambda}^{k+h}$. Let

$$\nu = \prod_{i=1}^{h} [B_{k+i}, A_{k+i}] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \delta = \prod_{i=1}^{k} [A_i, B_i] = \xi_\lambda \nu = \begin{pmatrix} \lambda a & \lambda b \\ \lambda^{-1} c & \lambda^{-1} d \end{pmatrix}.$$

Note that $t_1 = \operatorname{tr} \nu$, $t_2 = \operatorname{tr} \delta$ and λ determine a, d, and $bc = ad - 1$.

We follow the stratification in terms of the traces (t_1, t_2) given in [53, Section 10] for the genus 2 case. We decompose $\overline{X}_{4,\lambda} = \bigsqcup_{j=1}^{7} Z_{j,\lambda}$, where

- $Z_{1,\lambda}$ corresponding to $t_1 = \pm 2, t_2 = \pm 2$. In this case both ν, δ are of Jordan type. We focus on the case $(t_1, t_2) = (2, 2)$, the other cases being similar. Taking an adequate basis,
$$\nu = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad \delta = \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix},$$
for certain $x, y \in \mathbb{C}^*$. We can fix $x = 1$ by rescaling the basis vectors. Since $\delta\nu^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, we obtain that $\lambda + \lambda^{-1} = 2 - xy$, so y is also fixed. When varying λ, we see that there is no monodromy around the punctures. Therefore, taking also care of all four possibilities for (t_1, t_2), we have
$$\begin{aligned} R(Z_1) &= e(Z_{1,\lambda})T \\ &= (q-1)(e(\overline{X}_2^k)e(\overline{X}_2^h) + e(\overline{X}_2^k)e(\overline{X}_3^h) + e(\overline{X}_3^k)e(\overline{X}_2^h) + e(\overline{X}_3^k)e(\overline{X}_3^h))T \\ &= (q-1)(e(\overline{X}_2^k) + e(\overline{X}_3^k))(e(\overline{X}_2^h) + e(\overline{X}_3^h))T, \end{aligned}$$
where T is the trivial representation.

- $Z_{2,\lambda}$ corresponding to $(t_1, t_2) = (2, \lambda + \lambda^{-1})$ and $(t_1, t_2) = (-2, -\lambda - \lambda^{-1})$. We focus on the first case. In this situation $bc = 0$, so there are three possibilities: either $b = c = 0$ (in which case $\nu = \mathrm{Id}$) or $b = 0, c \neq 0$ or $b \neq 0, c = 0$ (in either case there is a parameter in $\mathbb{C}^*$ and $\nu \sim J_+$). In all cases, there is no monodromy for ν as λ moves in $\mathbb{C} - \{0, \pm 1\}$. On the other hand, $\delta \sim \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ gives a contribution $R(\overline{X}_4^h)$. The second case is analogous, changing Id by $-\mathrm{Id}$ and J_+ by J_-. Therefore
$$R(Z_2) = (e(\overline{X}_0^k) + 2(q-1)e(\overline{X}_2^k) + e(\overline{X}_1^k) + 2(q-1)e(\overline{X}_3^k))R(\overline{X}_4^h).$$

- $Z_{3,\lambda}$ corresponding to $(t_1, t_2) = (\lambda + \lambda^{-1}, 2)$ and $(t_1, t_2) = (-\lambda - \lambda^{-1}, -2)$. This is completely analogous to the previous case, so
$$R(Z_3) = (e(\overline{X}_0^h) + 2(q-1)e(\overline{X}_2^h) + e(\overline{X}_1^h) + 2(q-1)e(\overline{X}_3^h))R(\overline{X}_4^k).$$

- $Z_{4,\lambda}$ defined by $t_1 = 2, t_2 \neq \pm 2, \lambda + \lambda^{-1}$ and $t_1 = -2, t_2 \neq \pm 2, -\lambda - \lambda^{-1}$. For each λ, (t_1, t_2) move in (two) punctured lines $\{(t_1, t_2) \mid t_1 = \pm 2, t_2 \neq \pm 2, \pm(\lambda + \lambda^{-1})\}$, where ν is of Jordan type and δ is of diagonal type. Both families can be trivialized, giving a contribution of $e(\overline{X}_2^k)$ times $e(\overline{X}_4^h/\mathbb{Z}_2)$ for one line, and $e(\overline{X}_3^k)$ times $e(\overline{X}_4^h/\mathbb{Z}_2)$ for the other line. The missing fiber $\overline{X}_{4,\lambda}^h$ over $\lambda + \lambda^{-1}$, which needs to be removed, has monodromy given by $R(\overline{X}_4^h)$ as λ varies. Finally, there is a $(q-1)$ factor due to the fact that $bc \neq 0$. Therefore

$$R(Z_4) = (q-1)(e(\overline{X}_2^k) + e(\overline{X}_3^k))(e(\overline{X}_4^h/\mathbb{Z}_2)T - R(\overline{X}_4^h)).$$

- $Z_{5,\lambda}$ defined by $t_2 = 2, t_1 \neq \pm 2, \lambda + \lambda^{-1}$ and $t_2 = -2, t_1 \neq \pm 2, -\lambda - \lambda^{-1}$. Similarly to $Z_{4,\lambda}$, we obtain

$$R(Z_5) = (q-1)(e(\overline{X}_2^h) + e(\overline{X}_3^h))(e(\overline{X}_4^k/\mathbb{Z}_2)T - R(\overline{X}_4^k)).$$

- $Z_{6,\lambda}$. This stratum corresponds to the set $\{(t_1, t_2) \mid t_1, t_2 \neq \pm 2, ad = 1\}$, which is a hyperbola H_λ for every λ (see [53, Figure 1, Section 10]). There is a contribution of $2q - 1$ which accounts for $bc = 0$. Parametrizing H_λ by $\mu \in \mathbb{C}^* - \{\pm 1, \pm \lambda^{-1}\}$ as in [53, Section 10], we obtain a fibration over $\mathbb{C}^* - \{\pm 1, \pm\lambda^{-1}\}$ whose fiber over μ is $\overline{X}_{4,\mu}^k \times \overline{X}_{4,\lambda\mu}^h$, for each λ. When λ varies over $\mathbb{C} - \{0, \pm 1\}$, we can extend the local system trivially to the cases $\lambda, \mu = \pm 1$. This extension can be regarded as a local system over the set of $(\lambda, \mu) \in \mathbb{C}^* \times \mathbb{C}^*$,

$$\overline{Z}_6 = \overline{X}_4^k \times m^* \overline{X}_4^h \longrightarrow \mathbb{C}^* \times \mathbb{C}^*,$$

where $m : \mathbb{C}^* \times \mathbb{C}^* \to \mathbb{C}^*$ maps $(\lambda, \mu) \mapsto \lambda\mu$. The Hodge monodromy representation of $\overline{Z}_6$ belongs to $R(\mathbb{Z}_2 \times \mathbb{Z}_2)[q]$ (with generators N_1, N_2 denoting the representation which is not trivial over the generator of the fundamental group of the first and second copies of $\mathbb{C}^*$, respectively, and $N_{12} = N_1 \otimes N_2$). So we get

$$\begin{aligned} R_{\mathbb{C}^* \times \mathbb{C}^*}(\overline{Z}_6) &= ((a_k + d_k)T + (b_k + c_k)N_2) \otimes ((a_h + d_h)T + (b_h + c_h)N_{12}) \\ &= (a_h + d_h)(a_k + d_k)T + (b_h + c_h)(b_k + c_k)N_1 \\ &+ (a_h + d_h)(b_k + c_k)N_2 + (b_h + c_h)(a_k + d_k)N_{12}. \end{aligned}$$

To obtain the Hodge monodromy representation over $\lambda \in \mathbb{C}^*$, we use the projection $\pi_1 : \mathbb{C}^* \times \mathbb{C}^* \to \mathbb{C}^*$, $(\lambda, \mu) \mapsto \lambda$, which maps $T \mapsto e(T)T = (q-1)T$, $N_2 \mapsto e(N_2)T = 0$, $N_1 \mapsto e(T)N = (q-1)N$, $N_{12} \mapsto e(N_2)N = 0$ for the representations. Therefore $R_{\mathbb{C}^*}(\overline{Z}_6) = (q-1)((a_h+d_h)(a_k+d_k)T + (b_h+c_h)(b_k+c_k)N)$. Now we have to substract the contribution from the sets $\mu = \pm 1, \pm\lambda^{-1}$. The first two yield $-2e(\overline{X}_4^k)R(\overline{X}_4^h)$ and the second two yield $-2e(\overline{X}_4^h)R(\overline{X}_4^k)$. Therefore

$$\begin{aligned} R(\overline{Z}_6) =&(q-1)((a_h+d_h)(a_k+d_k)T + (b_h+c_h)(b_k+c_k)N) \\ &- 2e(\overline{X}_{4,\lambda}^k)R(\overline{X}_4^h) - 2e(\overline{X}_{4,\lambda}^h)R(\overline{X}_4^k), \\ R(Z_6) =&(2q-1)R(\overline{Z}_6). \end{aligned}$$

- $Z_{7,\lambda}$ corresponding to the open stratum given by the set of (t_1, t_2) such that $t_i \neq \pm 2$, $i = 1, 2$ and $(t_1, t_2) \notin H_\lambda$. If we forget about the condition $(t_1, t_2) \in H_\lambda$, $Z_{7,\lambda}$ is a fibration over (t_1, t_2) with fiber isomorphic to $\overline{X}_{4,\mu_1}^h \times \overline{X}_{4,\mu_2}^k$, $t_i = \mu_i + \mu_i^{-1}$, $i = 1, 2$. Its monodromy is trivial, as the local system is trivial when λ varies. The contribution over H_λ, already computed in the previous stratum, is $R(\overline{Z}_6)$. So we get

$$R(Z_7) =(q-1)(e(\overline{X}_4^k/\mathbb{Z}_2)e(\overline{X}_4^h/\mathbb{Z}_2)T - R(\overline{Z}_6)).$$

Adding all the pieces, we get

$$\begin{aligned} R(\overline{X}_4^{k+h}) = &(q-1)(e(\overline{X}_2^k) + e(\overline{X}_3^k) + e(\overline{X}_4^k/\mathbb{Z}_2))(e(\overline{X}_2^h) + e(\overline{X}_3^h) + e(\overline{X}_4^h/\mathbb{Z}_2))T \\ &+ (e(\overline{X}_0^k) + e(\overline{X}_1^k) + (q-1)e(\overline{X}_2^k) + (q-1)e(\overline{X}_3^k))R(\overline{X}_4^h) \\ &+ (e(\overline{X}_0^h) + e(\overline{X}_1^h) + (q-1)e(\overline{X}_2^h) + (q-1)e(\overline{X}_3^h))R(\overline{X}_4^k) + qR(\overline{Z}_6). \end{aligned}$$

Setting $k = g-1$, $h = 1$, we have:

$$\begin{aligned} R(\overline{X}_4^g) = \Big(&(q^3-1)e_0^{g-1} + (q^3-1)e_1^{g-1} + (q^5-3q^3+2q^2)e_2^{g-1} \\ &+ (q^5-3q^3+2q^2)e_3^{g-1} + (q^6-2q^5-2q^4+4q^3-3q^2+2)a_{g-1} \\ &+ (-q^5-q^4+2q^3-2q^2+q+1)(b_{g-1}+c_{g-1}) + (-q^4-2q^2+2q+1)d_{g-1}\Big)T \\ &+ \Big((3q^2-3q)e_0^{g-1} + (3q^2-3q)e_1^{g-1} + (3q^3-6q^2+3q)e_2^{g-1} + (3q^3-6q^2+3q)e_3^{g-1} \\ &+ (-6q^3+6q^2)(a_{g-1}+d_{g-1}) + (4q^4-14q^3+10q^2)(b_{g-1}+c_{g-1})\Big)N. \end{aligned} \tag{5.9}$$

5.8 Computation of $R(\overline{X}_4^g/\mathbb{Z}_2)$

Lemma 5.8.1. *Suppose that* $R(\overline{X}_4^k/\mathbb{Z}_2) = a_kT + b_kS_2 + c_kS_{-2} + d_kS_0$*, for all* $k < g$*. Then the Hodge monodromy representation* $R(\overline{X}_4^g/\mathbb{Z}_2)$ *is of the form* $R(\overline{X}_4^g/\mathbb{Z}_2) = a_gT + b_gS_2 + c_gS_{-2} + d_gS_0$*, for some polynomials* $a_g, b_g, c_g, d_g \in \mathbb{Z}[q]$*.*

Proof. The Hodge monodromy representation $R(\overline{X}_4^g/\mathbb{Z}_2)$ lies in the representation ring of the fundamental group of $\mathbb{C} - \{\pm 2\}$. Under the double cover $\mathbb{C} - \{0, \pm 1\} \to \mathbb{C} - \{\pm 2\}$, it reduces to $R(\overline{X}_4^g)$. By Section 5.7, $R(\overline{X}_4^g)$ is of order 2. Hence $R(\overline{X}_4^g/\mathbb{Z}_2)$ has only monodromy of order 2 over the loops $\gamma_{\pm 2}$ around the points ± 2. This is the statement of the lemma. ☐

Proposition 5.8.2. *All $\overline{X}_0^g$, $\overline{X}_1^g$, $\overline{X}_2^g$, $\overline{X}_3^g$, $\overline{X}_4^g$ and $\overline{X}_{4,\lambda}^g$ are of balanced type.*

Proof. By [53, Proposition 2.8], if $Z = \bigsqcup Z_i$ and all Z_i are of balanced type, then Z is of balanced type. Also, if $\mathbb{Z}_2$ acts on Z and Z is of balanced type, so is $Z/\mathbb{Z}_2$. Also Theorem 2.3.2 says that if $F \to Z \to B$ is a fibration with F of balanced type, with either $B = \mathbb{C} - \{0, \pm 1\}$ and Hodge monodromy $R(Z) = aT + bN$ or $B = \mathbb{C} - \{\pm 2\}$ and Hodge monodromy $R(Z) = aT + bS_2 + cS_{-2} + dS_0$, then Z is of balanced type.

In [53] it is proved that the result holds for $g = 1, 2$. Also $\mathrm{SL}(2,\mathbb{C})$, $\mathrm{PGL}(2,\mathbb{C})$, $\mathrm{PGL}(2,\mathbb{C})/D$, $\mathrm{PGL}(2,\mathbb{C})/U$ are of balanced type. Now assume that all $\overline{X}_0^k$, $\overline{X}_1^k$, $\overline{X}_2^k$, $\overline{X}_3^k$, $\overline{X}_4^k$ and $\overline{X}_{4,\lambda}^k$ are of balanced type for $k < g$. A look at the description of all strata for which we compute the E-polynomials in Sections 5.3–5.6 convinces us that $\overline{X}_0^g$, $\overline{X}_1^g$, $\overline{X}_2^g$, $\overline{X}_3^g$ are of balanced type. The same is true for $\overline{X}_{4,\lambda}^g$ by the stratification in Section 5.7. Finally formula (5.9) gives us that $\overline{X}_4^g$ is also of balanced type. □

Now to find the four polynomials $a_g, b_g, c_g, d_g \in \mathbb{Z}[q]$, we need four equations. Two come from the fact that $R(\overline{X}_4^g) = (a_g + d_g)T + (b_g + c_g)N$. From (5.9), we have

$$\begin{aligned} a_g + d_g =& (q^3-1)e_0^{g-1} + (q^3-1)e_1^{g-1} + (q^5 - 3q^3 + 2q^2)e_2^{g-1} \\ &+ (q^5 - 3q^3 + 2q^2)e_3^{g-1} + (q^6 - 2q^5 - 2q^4 + 4q^3 - 3q^2 + 2)a_{g-1} \\ &+ (-q^5 - q^4 + 2q^3 - 2q^2 + q + 1)(b_{g-1} + c_{g-1}) + (-q^4 - 2q^2 + 2q + 1)d_{g-1}. \end{aligned} \tag{5.10}$$

$$\begin{aligned} b_g + d_g =& (3q^2 - 3q)e_0^{g-1} + (3q^2 - 3q)e_1^{g-1} + (3q^3 - 6q^2 + 3q)e_2^{g-1} + (3q^3 - 6q^2 + 3q)e_3^{g-1} \\ &+ (-6q^3 + 6q^2)(a_{g-1} + d_{g-1}) + (4q^4 - 14q^3 + 10q^2)(b_{g-1} + c_{g-1}). \end{aligned} \tag{5.11}$$

One more equation is obtained by computing $e(\overline{X}_0^{g+1})$ with $k = g$, $h = 1$ and with $k = g-1$, $h = 2$, and equating. From (α) with $k = g$, we get

$$\begin{aligned} e_0^{g+1} =& (q^4 + 4q^3 - q^2 - 4q)e_0^g + (q^3 - q)e_1^g \\ &+ (q^5 - 2q^4 - 4q^3 + 2q^2 + 3q)e_2^g + (q^5 + 3q^4 - q^3 - 3q^2)e_3^g \\ &+ (q^6 - 2q^5 - 4q^4 + 3q^2 + 2q)a_g + (-q^5 - 4q^4 + 4q^2 + q)b_g \\ &+ (2q^5 - 7q^4 - 3q^3 + 7q^2 + q)c_g + (-5q^4 - q^3 + 5q^2 + q)d_g \end{aligned} \tag{5.12}$$

Using (5.5) with $k = g-1$, $h = 2$, and the values of $e(\overline{X}_j^2)$ and a_2, b_2, c_2, d_2 from Section

5.2, we have

$$\begin{aligned} e_0^{g+1} = &(q^9 + q^8 + 12q^7 + 2q^6 - 3q^4 - 12q^3 - q)e_0^{g-1} + (q^9 - 3q^7 - 30q^6 + 30q^4 + 3q^3 - q)e_1^{g-1} \\ &+ (q^{11} - 4q^9 - 4q^8 - 36q^7 + 24q^5 + 4q^4 + 15q^3)e_2^{g-1} \\ &+ (q^{11} - 4q^9 + 15q^8 + 9q^7 + 30q^6 - 6q^5 - 45q^4)e_3^{g-1} \\ &+ (q^{12} - 2q^{11} - 4q^{10} + 6q^9 - 6q^8 + 18q^7 - 6q^6 - 18q^5 + 15q^4 - 6q^3 + 2q)a_{g-1} \\ &+ (-q^{11} - q^{10} + 3q^9 - 42q^8 + 54q^7 + 30q^6 - 54q^5 + 12q^4 - 3q^3 + q^2 + q)b_{g-1} \\ &+ (-q^{11} - q^{10} + 18q^9 - 27q^8 + 24q^7 - 39q^5 + 27q^4 - 3q^3 + q^2 + q)c_{g-1} \\ &+ (-2q^{10} - 9q^8 + 24q^7 - 21q^5 + 9q^4 - 4q^3 + 2q^2 + q)d_{g-1} \end{aligned} \tag{5.13}$$

A fourth equation are obtained by computing $e(\overline{X}_1^{g+1})$ with $k = g$, $h = 1$ and with $k = g - 1$, $h = 2$, and equating. From (β) with $k = g$, we get

$$\begin{aligned} e_1^{g+1} = &(q^3 - q)e_0^g + (q^4 + 4q^3 - q^2 - 4q)e_1^g \\ &+ (q^5 + 3q^4 - q^3 - 3q^2)e_2^g + (q^5 - 2q^4 - 4q^3 + 2q^2 + 3q)e_3^g \\ &+ (q^6 - 2q^5 - 4q^4 + 3q^2 + 2q)a_g + (2q^5 - 7q^4 - 3q^3 + 7q^2 + q)b_g \\ &+ (-q^5 - 4q^4 + 4q^2 + q)c_g + (-5q^4 - q^3 + 5q^2 + q)d_g \end{aligned} \tag{5.14}$$

Using (5.8) with $k = g - 1$, $h = 2$, and the values of $e(\overline{X}_j^2)$ and a_2, b_2, c_2, d_2 from Section 5.2, we have

$$\begin{aligned} e_1^{g+1} = &(q^9 - 3q^7 - 30q^6 + 30q^4 + 3q^3 - q)e_0^{g-1} + (q^9 + q^8 + 12q^7 + 2q^6 - 3q^4 - 12q^3 - q)e_1^{g-1} \\ &+ (q^{11} - 4q^9 + 15q^8 + 9q^7 + 30q^6 - 6q^5 - 45q^4)e_2^{g-1} \\ &+ (q^{11} - 4q^9 - 4q^8 - 36q^7 + 24q^5 + 4q^4 + 15q^3)e_3^{g-1} \\ &+ (q^{12} - 2q^{11} - 4q^{10} + 6q^9 - 6q^8 + 18q^7 - 6q^6 - 18q^5 + 15q^4 - 6q^3 + 2q)a_{g-1} \\ &+ (-q^{11} - q^{10} + 18q^9 - 27q^8 + 24q^7 - 39q^5 + 27q^4 - 3q^3 + q^2 + q)c_{g-1} \\ &+ (-q^{11} - q^{10} + 3q^9 - 42q^8 + 54q^7 + 30q^6 - 54q^5 + 12q^4 - 3q^3 + q^2 + q)b_{g-1} \\ &+ (-2q^{10} - 9q^8 + 24q^7 - 21q^5 + 9q^4 - 4q^3 + 2q^2 + q)d_{g-1} \end{aligned} \tag{5.15}$$

The solutions to (5.10), (5.11), (5.12)=(5.13) and (5.14)=(5.15), and using the values of (α), (β), (γ) and (δ), are given by

$$\begin{aligned} a_g = &q^3 e_0^{g-1} + q^3 e_1^{g-1} + (q^5 - 3q^3)e_2^{g-1} + (q^5 - 3q^3)e_3^{g-1} + (q^6 - 2q^5 - 2q^4 + 4q^3 + q^2)a_{g-1} \\ &+ (-q^5 - q^4 + 2q^3)b_{g-1} + (-q^5 - q^4 + 2q^3)c_{g-1} - 2q^4 d_{g-1} \\ b_g = &-3qe_0^{g-1} + 3q^2 e_1^{g-1} + (3q^3 + 3q)e_2^{g-1} - 6q^2 e_3^{g-1} \\ &+ (-3q^3 + 3q^2)a_{g-1} + (4q^4 - 6q^3 + 4q^2)b_{g-1} + (-8q^3 + 6q^2)c_{g-1} + (q^2 - 3q^3 + 3q^2)d_{g-1} \\ c_g = &3q^2 e_0^{g-1} - 3qe_1^{g-1} - 6q^2 e_2^{g-1} + (3q^3 + 3q)e_3^{g-1} \end{aligned}$$

$$
\begin{aligned}
&\quad (-3q^3+3q^2)a_{g-1}+(-8q^3+6q^2)b_{g-1}+(4q^4-6q^3+4q^2)c_{g-1}+(-3q^3+3q^2)d_{g-1}\\
d_g =&-e_0^{g-1}-e_1^{g-1}+2q^2e_2^{g-1}+2q^2e_3^{g-1}\\
&+(-4q^2+2)a_{g-1}+(-2q^2+q+1)b_{g-1}+(-2q^2+q+1)c_{g-1}+(q^4-2q^2+2q+1)d_{g-1}
\end{aligned}
$$

We put this together with equations (α), (β), (γ) and (δ)

$$
\begin{aligned}
e_0^g =&(q^4+4q^3-q^2-4q)e_0^{g-1}+(q^3-q)e_1^{g-1}\\
&+(q^5-2q^4-4q^3+2q^2+3q)e_2^{g-1}+(q^5+3q^4-q^3-3q^2)e_3^{g-1}\\
&+(q^6-2q^5-4q^4+3q^2+2q)a_{g-1}+(-q^5-4q^4+4q^2+q)b_{g-1}\\
&+(2q^5-7q^4-3q^3+7q^2-q)c_{g-1}+(-5q^4-q^3+5q^2-q)d_{g-1}.\\
e_1^g =&(q^3-q)e_0^{g-1}+(q^4+4q^3-q^2-4q)e_1^{g-1}\\
&+(q^5+3q^4-q^3-3q^2)e_2^{g-1}+(q^5-2q^4-4q^3+2q^2+3q)e_3^{g-1}\\
&+(q^6-2q^5-4q^4+3q^2+2q)a_{g-1}+(2q^5-7q^4-3q^3+7q^2+q)b_{g-1}\\
&+(-q^5-4q^4+4q^2+q)c_{g-1}+(-5q^4-q^3+5q^2+q)d_{g-1}.\\
e_2^g =&(q^3-2q^2-3q)e_0^{g-1}+(q^3+3q^2)e_1^{g-1}\\
&+(q^5+q^4+3q^2+3q)e_2^{g-1}+(q^5-3q^3-6q^2)e_3^{g-1}\\
&+(q^6-2q^5-3q^4+q^3+3q^2)a_{g-1}+(-q^5+2q^4-4q^3+3q^2)b_{g-1}\\
&+(-q^5-q^4-4q^3+6q^2)c_{g-1}+(-2q^4-q^3+3q^2)d_{g-1}.\\
e_3^g =&(q^3+3q^2)e_0^{g-1}+(q^3-2q^2-3q)e_1^{g-1}\\
&+(q^5-3q^3-6q^2)e_2^{g-1}+(q^5+q^4+3q^2+3q)e_3^{g-1}\\
&+(q^6-2q^5-3q^4+q^3+3q^2)a_{g-1}+(-q^5-q^4-4q^3+6q^2)b_{g-1}\\
&+(-q^5+2q^4-4q^3+3q^2)c_{g-1}+(-2q^4-q^3+3q^2)d_{g-1}.
\end{aligned}
$$

Hence there is a 8×8-matrix M such that if we write $v_g=(e_0^g,e_1^g,e_2^g,e_3^g,a_g,b_g,c_g,d_g)^t$,

$$
v_g = Mv_{g-1}, \tag{5.16}
$$

for all $g \geq 3$. M is the following matrix

$$\begin{pmatrix}
\substack{q^4+4q^3\\-q^2-4q} & q^3-q & \substack{q^5-2q^4-4q^3\\+2q^2+3q} & \substack{q^5+3q^4\\-q^3-3q^2} & \substack{q^6-2q^5-4q^4\\+3q^2+2q} & \substack{-q^5-4q^4\\+4q^2+q} & \substack{2q^5-7q^4-3q^3\\+7q^2+q} & \substack{-5q^4-q^3\\+5q^2+q} \\
q^3-q & \substack{q^4+4q^3\\-q^2-4q} & \substack{q^5+3q^4\\-q^3-3q^2} & \substack{q^5-2q^4-4q^3\\+2q^2+3q} & \substack{q^6-2q^5-4q^4\\+3q^2+2q} & \substack{2q^5-7q^4-3q^3\\+7q^2+q} & \substack{-q^5-4q^4\\+4q^2+q} & \substack{-5q^4-q^3\\+5q^2+q} \\
\substack{q^3-2q^2\\-3q} & q^3+3q^2 & \substack{q^5+q^4\\+3q^2+3q} & \substack{q^5-3q^3\\-6q^2} & \substack{q^6-2q^5-3q^4\\+q^3+3q^2} & \substack{-q^5+2q^4\\-4q^3+3q^2} & \substack{-q^5-q^4\\-4q^3+6q^2} & \substack{-2q^4-q^3\\+3q^2} \\
q^3+3q^2 & \substack{q^3-2q^2\\-3q} & \substack{q^5-3q^3\\-6q^2} & \substack{q^5+q^4\\+3q^2+3q} & \substack{q^6-2q^5-3q^4\\+q^3+3q^2} & \substack{-q^5-q^4\\-4q^3+6q^2} & \substack{-q^5+2q^4\\-4q^3+3q^2} & \substack{-2q^4-q^3\\+3q^2} \\
q^3 & q^3 & q^5-3q^3 & q^5-3q^3 & \substack{q^6-2q^5-2q^4\\+4q^3+q^2} & \substack{-q^5-q^4\\+2q^3} & \substack{-q^5-q^4\\+2q^3} & -2q^4 \\
-3q & 3q^2 & 3q^2+3 & -6q^2 & -3q^3+3q^2 & 4q^4-6q^3+4q^2 & -8q^3+6q^2 & -3q^3+3q^2 \\
3q^2 & -3q & -6q^2 & 3q^3+3q & -3q^3+3q^2 & -8q^3+6q^2 & 4q^4-6q^3+4q^2 & -3q^3+3q^2 \\
-1 & -1 & 2q^2 & 2q^2 & -4q^2+2 & -2q^2+q+1 & -2q^2+q+1 & \substack{q^4-2q^2\\+2q+1}
\end{pmatrix}. \tag{5.17}$$

The starting vector is given in Section 5.2, $v_2 = (e_0^2, e_1^2, e_2^2, e_3^2, a_2, b_2, c_2, d_2)^t = (q^9 + q^8 + 12q^7 + 2q^6 - 3q^4 - 12q^3 - q, q^9 - 3q^7 - 30q^6 + 30q^4 + 3q^3 - q, q^9 - 3q^7 - 4q^6 - 39q^5 - 4q^4 - 15q^3, q^9 - 3q^7 + 15q^6 + 6q^5 + 45q^4, q^9 - 3q^7 + 6q^5, -(45q^5 + 15q^3), 15q^6 + 45q^4, -6q^4 + 3q^2 - 1)^t$. If we write $v_1 = (e_0^1, e_1^1, e_2^1, e_3^1, a_1, b_1, c_1, d_1)^t = (q^4 + 4q^3 - q^2 - 4q, q^3 - q, q^3 - 2q^2 - 3q, q^3 + 3q^2, q^3, -3q, 3q^2, -1)^t$ and

$$v_0 = (1, 0, 0, 0, 0, 0, 0, 0)^t,$$

then equation (5.16) holds for all $g \geq 1$. So

$$v_g = M^g v_0.$$

Remark 5.8.3. *As in [53], we can stratify* $SL(2, \mathbb{C})^{2g} = \bigsqcup_{i=0}^4 X_i^g$, *with*

- $X_0^g = \overline{X}_0^g$, $e(X_0^g) = e_0^g$.
- $X_1^g = \overline{X}_1^g$, $e(X_1^g) = e_1^g$.
- $X_2^g = \{(A_1, B_1, \ldots, A_g, B_g) \mid \prod_{i=1}^g [A_i, B_i] \sim J_+\} \cong (\mathrm{PGL}(2, \mathbb{C})/U) \times \overline{X}_2^g$. *So* $e(X_2^g) = (q^2 - 1)e_2^g$.
- $X_3^g = \{(A_1, B_1, \ldots, A_g, B_g) \mid \prod_{i=1}^g [A_i, B_i] \sim J_-\} \cong (\mathrm{PGL}(2, \mathbb{C})/U) \times \overline{X}_3^g$. *So* $e(X_3^g) = (q^2 - 1)e_3^g$.
- $X_4^g = \{(A_1, B_1, \ldots, A_g, B_g) \mid \prod_{i=1}^g [A_i, B_i] \sim \xi_\lambda$, *for some* $\lambda \in \mathbb{C} - \{0, \pm 1\}\}$. *Here* $X_4^g \cong (\mathrm{PGL}(2, \mathbb{C})/D \times \overline{X}_4^g)/\mathbb{Z}_2$. *Using (5.4), we have* $e(X_4^g) = (q^3 - 2q^2 - q)a_g - (q^2 + q)(b_g + c_g) - 2qd_g$.

Therefore it must be

$$(q^3-q)^{2g} = e_0^g + e_1^g + (q^2-1)(e_2^g+e_3^g) + (q^3-2q^2-q)a_g - (q^2+q)(b_g+c_g) - 2qd_g. \quad (5.18)$$

We can prove (5.18) numerically by induction on $g \geq 0$, using (5.16). The equation (5.18) is certainly true for $g=0$. Suppose it holds for $g-1$ and let $w = (w_0, \dots, w_7)^t = Mv_{g-1}$. Then an easy computation gives

$$\begin{aligned}
&w_0 + w_1 + (q^2-1)(w_2+w_3) + (q^3-2q^2-q)w_4 - (q^2+q)(w_5+w_6) - 2qw_7 \\
&\quad = (q^3-q)^2(e_0^{g-1} + e_1^{g-1} + (q^2-1)(e_2^{g-1}+e_3^{g-1}) \\
&\quad + (q^3-2q^2-q)a_{g-1} - (q^2+q)(b_{g-1}+c_{g-1}) - 2qd_{g-1}) \\
&\quad = (q^3-q)^2(q^3-q)^{2g-2} = (q^3-q)^{2g},
\end{aligned}$$

so equation (5.18) holds for $v_g = w = Mv_{g-1}$.

We start by proving Corollary 5.1.3 using (5.17).

Theorem 5.8.4. *For every $g \geq 1$, we have $e(\mathcal{M}_{J_-}) + (q+1)e(\mathcal{M}_{-\mathrm{Id}}) = e(\mathcal{M}_{\xi_\lambda})$.*

Proof. First, $\mathcal{M}_{-\mathrm{Id}} = \overline{X}_1^g / \mathrm{PGL}(2,\mathbb{C})$, so $e(\mathcal{M}_{-\mathrm{Id}}) = e_1^g/(q^3-q)$. Second, $\mathcal{M}_{J_+} = \overline{X}_2^g/U$, so $e(\mathcal{M}_{J_+}) = e_3^g/q$. And third, $\mathcal{M}_{\xi_\lambda} = \overline{X}_{4,\lambda}^g/D$, so $e(\mathcal{M}_{\xi_\lambda}) = e(\overline{X}_{4,\lambda}^g)/(q-1) = (a_g+b_g+c_g+d_g)/(q-1)$.

The assertion is thus equivalent to

$$(q^2-1)e_3^g + (q+1)e_1^g = (q^2+q)(a_g+b_g+c_g+d_g), \quad (5.19)$$

for all $g \geq 1$. We proceed by induction starting with $g=0$, where it obviously holds. If we assume that (5.19) holds for $g-1$, then using (5.17),

$$\begin{aligned}
&(q^2+q)(a_g+b_g+c_g+d_g) - (q^2-1)e_3^g - (q+1)e_1^g \\
&\quad = -q^2(q+1)(q-1)^2 e_1^{g-1} - q^2(q+1)(q-1)^3 e_3^{g-1} \\
&\quad + q^3(q+1)(q-1)^2(a_{g-1}+b_{g-1}+c_{g-1}+d_{g-1}) \\
&\quad = q^2(q-1)^2((q^2+q)(a_{g-1}+b_{g-1}+c_{g-1}+d_{g-1}) - (q^2-1)e_3^{g-1} - (q+1)e_1^{g-1}) = 0,
\end{aligned}$$

by induction hypothesis. □

Since $v_g = M^g v_0$, we can obtain closed formulas for $e_0^g, e_1^g, e_2^g, e_3^g, a_g, b_g, c_g, d_g$. We summarize them in the following

Proposition 5.8.5. *For all $g \geq 1$,*

$$\begin{aligned}
e_0^g &= (q^3-q)\Bigg((q^3-q)^{2g-2} + (q^2-1)^{2g-2} - (q^2-q)^{2g-2} \\
&\quad + \frac{1}{2}q^{2g-2}(q+2^{2g}-1)\left((q+1)^{2g-2}+(q-1)^{2g-2}\right)\Bigg), \\
e_1^g &= (q^3-q)\left((q^3-q)^{2g-2} + (q^2-1)^{2g-2} - 2^{2g-1}(q^2+q)^{2g-2} + (2^{2g-1}-1)(q^2-q)^{2g-2}\right), \\
e_2^g &= (q^3-q)^{2g-1} + (2^{2g-1}-1)(q^2-q)^{2g-1} - 2^{2g-1}(q^2+q)^{2g-1} \\
&\quad + \frac{1}{2}q^{2g-1}(q-1)\left((q-1)^{2g-1}-(q+1)^{2g-1}\right), \\
e_3^g &= (q^3-q)^{2g-1} + (2^{2g-1}-1)(q^2-q)^{2g-1} + 2^{2g-1}(q^2+q)^{2g-1}, \\
a_g &= (q^3-q)^{2g-1} + \frac{1}{2}q^{2g-1}\left((q+1)^{2g-1}-(q-1)^{2g-1}\right), \\
b_g &= 2^{2g-1}(q^2-q)^{2g-1} - 2^{2g-1}(q^2+q)^{2g-1} + \frac{1}{2}q^{2g-1}\left((q+1)^{2g-1}-(q-1)^{2g-1}\right), \\
c_g &= 2^{2g-1}(q^2-q)^{2g-1} + 2^{2g-1}(q^2+q)^{2g-1} - \frac{1}{2}q^{2g-1}\left((q+1)^{2g-1}+(q-1)^{2g-1}\right), \\
d_g &= (q^2-1)^{2g-1} - \frac{1}{2}q^{2g-1}\left((q+1)^{2g-1}+(q-1)^{2g-1}\right),
\end{aligned}$$

and also

$$e^g_{4,\xi_\lambda} = a_g + b_g + c_g + d_g = (q^3-q)^{2g-1} + (q^2-1)^{2g-1} + (2^{2g}-2)(q^2-q)^{2g-1}.$$

Proof. We know that $v_g = M^g v_0$, where M is given in 5.17. There exists a matrix Q with entries in the fraction field of $\mathbb{Z}[q]$ such that $M = QDQ^{-1}$, where D is the diagonal matrix

$$\begin{pmatrix}
(q^2-q)^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & (q^2+q)^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 4(q^2-q)^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 4(q^2+q)^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & (q^2-1)^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & (q^3-q)^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & (q^2-q)^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & (q^2+q)^2
\end{pmatrix}$$

As $M^g = QD^gQ^{-1}$, a straightforward computation gives the desired formulas. □

Proposition 5.8.5 gives us the E-polynomials of all the moduli spaces where the quotient is geometric dividing by the E-polynomials of the respective stabilizers. We obtain

Theorem 5.8.6. *For all $g \geq 1$,*

$$\begin{aligned}
e(\mathcal{M}_{-\mathrm{Id}}) &= e_1^g/(q^3-q) = (q^3-q)^{2g-2} + (q^2-1)^{2g-2} - 2^{2g-1}(q^2+q)^{2g-2} \\
&\quad + (2^{2g-1}-1)(q^2-q)^{2g-2} \\
e(\mathcal{M}_{J_+}) &= e_2^g/q = (q^3-q)^{2g-2}(q^2-1) + (2^{2g-1}-1)(q-1)(q^2-q)^{2g-2} \\
&\quad - 2^{2g-1}(q+1)(q^2+q)^{2g-2} + \frac{1}{2}q^{2g-2}(q-1)\left((q-1)^{2g-1}-(q+1)^{2g-1}\right)
\end{aligned}$$

$$\begin{aligned}e(\mathcal{M}_{J_-}) = & e_3^g/q = (q^3-q)^{2g-2}(q^2-1) + (2^{2g-1}-1)(q-1)(q^2-q)^{2g-2} \\ & + 2^{2g-1}(q+1)(q^2+q)^{2g-2} \\ e(\mathcal{M}_{\xi_\lambda}) = & e_{4,\xi_\lambda}^g/(q-1) = (q^3-q)^{2g-2}(q^2+q) + (q^2-1)^{2g-2}(q+1) \\ & + (2^{2g}-2)(q^2-q)^{2g-2}q.\end{aligned}$$

Note that $e(\mathcal{M}_{-\mathrm{Id}}^g)$ agrees with the result obtained by arithmetic methods in [60].

Corollary 5.8.7. *For $g \geq 1$, the behaviour of the E-polynomial of the parabolic character variety $\mathcal{M}_{\xi_\lambda}^g$ is given by*

$$\begin{aligned}R(\mathcal{M}_{\xi_\lambda}) = & \left((q^3-q)^{2g-2}(q^2+q) + (q+1)(q^2-1)^{2g-2} - q(q^2-q)^{2g-2}\right) T \\ & + \left((2^{2g}-1)q(q^2-q)^{2g-2}\right) N.\end{aligned}$$

Proof. From Proposition 5.8.5 we get that

$$\begin{aligned}R(\overline{X}_4^g) &= (a_g+d_g)T + (b_g+c_g)N \\ &= ((q^3-q)^{2g-1} + (q^2-1)^{2g-1} - (q^2-q)^{2g-1})T + ((2^{2g}-1)(q^2-q)^{2g-1})N.\end{aligned}$$

The result is obtained dividing by $e(\mathrm{Stab}\,(\xi_\lambda)) = q-1$. □

To complete the proof of Theorem 5.1.1, it remains the following

Theorem 5.8.8. *For all $g \geq 1$, we have*

$$\begin{aligned}e(\mathcal{M}_{\mathrm{Id}}) = & (q^3-q)^{2g-2} + (q^2-1)^{2g-2} - q(q^2-q)^{2g-2} - 2^{2g}q^{2g-2} \\ & + \frac{1}{2}q^{2g-2}(q+2^{2g}-1)((q+1)^{2g-2}+(q-1)^{2g-2}) + \frac{1}{2}q((q+1)^{2g-1}+(q-1)^{2g-1}).\end{aligned}$$

Proof. We need to distinguish between reducible and irreducible orbit since we have to take a GIT quotient to compute $e(\mathcal{M}_{\mathrm{Id}})$ and identify those orbits whose closures intersect. We compute the reducible locus, the E-polynomial of the irreducible locus is obtained by sustracting the contribution of the reducible part from the E-polynomial of the total space e_0^g.

A reducible representation given by $(A_1, B_1, A_2, B_2, \ldots, A_g, B_g) \in SL(2,\mathbb{C})^{2g}$ is S-equivalent to

$$\left(\begin{pmatrix}\lambda_1 & 0 \\ 0 & \lambda_1^{-1}\end{pmatrix}, \begin{pmatrix}\lambda_2 & 0 \\ 0 & \lambda_2^{-1}\end{pmatrix}, \ldots, \begin{pmatrix}\lambda_{2g} & 0 \\ 0 & \lambda_{2g}^{-1}\end{pmatrix}\right) \tag{5.20}$$

under the $\mathbb{Z}_2$-action $(\lambda_1, \lambda_2, \ldots, \lambda_{2g}) \sim (\lambda_1^{-1}, \lambda_2^{-1}, \ldots, \lambda_{2g}^{-1})$. We have that $e(\mathbb{C}^*)^+ = q$ and $e(\mathbb{C}^*)^- = -1$, so

$$\begin{aligned}e(\mathcal{M}_{\mathrm{Id}}^{red}) &= e((\mathbb{C}^*)^{2g}/\mathbb{Z}_2) \\ &= (e(\mathbb{C}^*)^+)^{2g} + \binom{2g}{2}(e(\mathbb{C}^*)^+)^{2g-2}(e(\mathbb{C}^*)^-)^2 + \ldots + (e(\mathbb{C}^*)^-)^{2g} \\ &= \frac{1}{2}\left((q-1)^{2g} + (q+1)^{2g}\right).\end{aligned}$$

A reducible representation occurs if there is a common eigenvector. With respect to a suitable basis, the representation takes the form

$$\left(\begin{pmatrix}\lambda_1 & a_1\\ 0 & \lambda_1^{-1}\end{pmatrix}, \begin{pmatrix}\lambda_2 & a_2\\ 0 & \lambda_2^{-1}\end{pmatrix}, \dots, \begin{pmatrix}\lambda_{2g} & a_{2g}\\ 0 & \lambda_{2g}^{-1}\end{pmatrix}\right),$$

which is a set parametrized by $(\mathbb{C}^* \times \mathbb{C})^{2g}$. The condition $\prod_{i=1}^{g}[A_i, B_i] = \mathrm{Id}$ is rewritten as

$$\sum_{i=1}^{g} \lambda_{2i}(\lambda_{2i-1}^2 - 1)a_{2i} - \lambda_{2i-1}(\lambda_{2i}^2 - 1)a_{2i-1} = 0. \tag{5.21}$$

There are four cases:

- R_1, given by $(a_1, a_2, \dots, a_{2g}) \in \langle(\lambda_1 - \lambda_1^{-1}, \lambda_2 - \lambda_2^{-1}, \dots, \lambda_{2g} - \lambda_{2g}^{-1})\rangle$, and the condition $(\lambda_1, \lambda_2, \dots, \lambda_{2g}) \neq (\pm 1, \pm 1, \dots, \pm 1)$. In this case we can conjugate the representation to the diagonal form (5.20) and assume that $a_i = 0$. The stabilizer of this stratum is the set of diagonal matrices $D \subset \mathrm{PGL}(2,\mathbb{C})$. Writing $A := (\mathbb{C}^*)^{2g} - \{(\pm 1, \pm 1, \dots, \pm 1)\}$, the stratum is isomorphic to $(A \times \mathrm{PGL}(2,\mathbb{C})/D)/\mathbb{Z}_2$, where the $\mathbb{Z}_2$-action is given by the permutation of the two basis vectors. Since $e(\mathrm{PGL}(2,\mathbb{C})/D)^+ = q^2, e(\mathrm{PGL}(2,\mathbb{C})/D)^- = q$ and

$$\begin{aligned} e(A)^+ &= \frac{1}{2}\left((q-1)^{2g} + (q+1)^{2g}\right) - 2^{2g} \\ e(A)^- &= e(A) - e(A)^+ = \frac{1}{2}\left((q-1)^{2g} - (q+1)^{2g}\right), \end{aligned}$$

  we obtain

$$\begin{aligned} e(R_1) &= e(\mathrm{PGL}(2,\mathbb{C})/D)^+ e(A)^+ + e(\mathrm{PGL}(2,\mathbb{C})/D)^- e(A)^- \\ &= q^2\left(\frac{1}{2}((q-1)^{2g} + (q+1)^{2g}) - 2^{2g}\right) + q\left(\frac{1}{2}((q-1)^{2g} - (q+1)^{2g})\right) \\ &= (q^3 - q)\frac{1}{2}\left((q-1)^{2g-1} + (q+1)^{2g-1}\right) - 2^{2g}q^2. \end{aligned}$$

- R_2, given by $(a_1, a_2, \dots, a_{2g}) \notin \langle(\lambda_1 - \lambda_1^{-1}, \lambda_2 - \lambda_2^{-1}, \dots, \lambda_{2g} - \lambda_{2g}^{-1})\rangle$, and the condition $(\lambda_1, \lambda_2, \dots, \lambda_{2g}) \neq (\pm 1, \pm 1, \dots, \pm 1)$. Equation (5.21) defines a hyperplane $H \subset \mathbb{C}^{2g}$ and the condition for $(a_1, a_2, \dots, a_{2g})$ defines a line $l \subset H$. Writing $U' \cong D \times U$ for the upper triangular matrices, we have a surjective map $A \times (H - l) \times \mathrm{PGL}(2,\mathbb{C}) \longrightarrow R_2$ with fiber isomorphic to U'. Hence

$$\begin{aligned} e(R_2) &= ((q-1)^{2g} - 2^{2g})(q^{2g-1} - q)(q^3 - q)/(q^2 - q) \\ &= (q+1)(q^{2g-1} - q)((q-1)^{2g} - 2^{2g}). \end{aligned}$$

- R_3, given by $(a_1, a_2, \ldots, a_{2g}) = (0, 0, \ldots, 0)$, and $(\lambda_1, \lambda_2, \ldots, \lambda_{2g}) = (\pm 1, \pm 1, \ldots, \pm 1)$, corresponding to the case where $A_i = B_i = \pm \mathrm{Id}$. The stratum consists of 2^{2g} points, so
$$e(R_3) = 2^{2g}.$$

- R_4, given by $(a_1, a_2, \ldots, a_{2g}) \neq (0, 0, \ldots, 0)$, and $(\lambda_1, \lambda_2, \ldots, \lambda_{2g}) = (\pm 1, \pm 1, \ldots, \pm 1)$. In this case, there is at least a matrix of Jordan type, so the diagonal matrices D act projectivizing the set $(a_1, a_2, \ldots, a_{2g}) \in \mathbb{C}^{2g} - \{(0, 0, \ldots, 0)\}$. The stabilizer is isomorphic to U. Therefore
$$\begin{aligned} e(R_4) &= 2^{2g} e(\mathbb{P}^{2g-1}) e(\mathrm{PGL}(2, \mathbb{C})/U) \\ &= 2^{2g}(q^{2g-1} + q^{2g-2} + \ldots + 1)(q^2 - 1) \\ &= 2^{2g}(q^{2g} - 1)(q + 1). \end{aligned}$$

The total E-polynomial of the reducible locus R is thus
$$\begin{aligned} e(R) &= e(R_1) + e(R_2) + e(R_3) + e(R_4) \\ &= (q^3 - q)\left(\frac{1}{2}((q+1)^{2g-1} - (q-1)^{2g-1}) + 2^{2g} q^{2g-2} + (q-1)(q^2 - q)^{2g-2}\right). \end{aligned}$$

We obtain the E-polynomial of the irreducible part as
$$\begin{aligned} e(I) =& e_0^g - e(R) \\ =& (q^3 - q)\Bigg((q^3 - q)^{2g-2} + (q^2 - 1)^{2g-2} - (q^2 - q)^{2g-2} + \frac{1}{2} q^{2g-2}(q + 2^{2g} - 1)((q+1)^{2g-2} \\ & + (q-1)^{2g-2}) - 2^{2g} q^{2g-2} - (q-1)(q^2 - q)^{2g-2} - \frac{1}{2}((q+1)^{2g-1} - (q-1)^{2g-1})\Bigg), \end{aligned}$$

and
$$\begin{aligned} e(\mathcal{M}_{\mathrm{Id}}^{irr}) =& e(I)/(q^3 - q) \\ =& (q^3 - q)^{2g-2} + (q^2 - 1)^{2g-2} - q(q^2 - q)^{2g-2} - 2^{2g} q^{2g-2} \\ & + \frac{1}{2} q^{2g-2}(q + 2^{2g} - 1)((q+1)^{2g-2} + (q-1)^{2g-2}) - \frac{1}{2}((q+1)^{2g-1} - (q-1)^{2g-1}). \end{aligned}$$

Finally,
$$\begin{aligned} e(\mathcal{M}_{\mathrm{Id}}) =& e(\mathcal{M}_{\mathrm{Id}}^{irr}) + e(\mathcal{M}_{\mathrm{Id}}^{red}) \\ =& (q^3 - q)^{2g-2} + (q^2 - 1)^{2g-2} - q(q^2 - q)^{2g-2} - 2^{2g} q^{2g-2} \\ & + \frac{1}{2} q^{2g-2}(q + 2^{2g} - 1)((q+1)^{2g-2} + (q-1)^{2g-2}) \\ & + \frac{1}{2} q((q+1)^{2g-1} + (q-1)^{2g-1}). \end{aligned}$$

□

5.9 Topological consequences

In this section, we extract some information from the formulas in Theorem 5.1.1. We start by a proof of Theorem 5.1.2.

Proof of Theorem 5.1.2. In the case of $\mathcal{M}_{-\mathrm{Id}}, \mathcal{M}_{J_\pm}$ and $\mathcal{M}_{\xi_\lambda}$, the result follows readily from Proposition 5.8.2. For instance, for $\mathcal{M}_{-\mathrm{Id}}$ we have that $\mathcal{M}_{-\mathrm{Id}} = \overline{X}_1/\operatorname{PGL}(2,\mathbb{C})$, where $\overline{X}_1$ and $G = \operatorname{PGL}(2,\mathbb{C})$ are of balanced type. Hence the classifying space BG is also of balanced type and there is a homotopy fibration $\overline{X}_1 \to \overline{X}_1/G \to BG$. The Leray spectral sequence gives that $\overline{X}_1/G$ must be of balanced type (a similar argument appears in the proof of Proposition 7.2 of [62]).

In the case of $\mathcal{M}_{\mathrm{Id}}$, the description in Theorem 5.8.8 yields that R is of balanced type. Hence I is also of balanced type. The same argument as above proves that $\mathcal{M}_{\mathrm{Id}}^{irr} = I/\operatorname{PGL}(2,\mathbb{C})$ is of balanced type. As $\mathcal{M}_{\mathrm{Id}}^{red}$ is clearly of balanced type, so is $\mathcal{M}_{\mathrm{Id}}$. □

Corollary 5.9.1. *Let X be a complex curve of genus $g \geq 2$. The Euler characteristic of $\mathcal{M}_C = \mathcal{M}_C(\mathrm{SL}(2,\mathbb{C}))$ is given by*

$$\begin{aligned}
\chi(\mathcal{M}_{\mathrm{Id}}) &= 2^{4g-3} - 3\cdot 2^{2g-2}\\
\chi(\mathcal{M}_{-\mathrm{Id}}) &= -2^{4g-3}\\
\chi(\mathcal{M}_{J_+}) &= -2^{4g-2}\\
\chi(\mathcal{M}_{J_-}) &= 2^{4g-2}\\
\chi(\mathcal{M}_{\xi_\lambda}) &= 0.
\end{aligned}$$

Proof. The Euler characteristic is obtained by setting $q = 1$ in $e(\mathcal{M}_C)$ given in Theorem 5.1.1. □

Corollary 5.9.2. *Let X be a complex curve of genus $g \geq 2$. Then $\mathcal{M}_{\mathrm{Id}}$ and $\mathcal{M}_{-\mathrm{Id}}$ are of dimension $6g-6$ and $\mathcal{M}_{J_+}$, $\mathcal{M}_{J_-}$ and $\mathcal{M}_{\xi_\lambda}$ are of dimension $6g-4$. All of them have a unique component of maximal dimension.*

Proof. From Theorem 5.1.1, we get

$$\begin{aligned}
e(\mathcal{M}_{\mathrm{Id}}) &= q^{6g-6} + \ldots + 1\\
e(\mathcal{M}_{-\mathrm{Id}}) &= q^{6g-6} + \ldots + 1\\
e(\mathcal{M}_{J_+}) &= q^{6g-4} + \ldots + (1 - 2^{2g-1})q^{2g-2}\\
e(\mathcal{M}_{J_-}) &= q^{6g-4} + \ldots + (2^{2g} - 1)q^{2g-1}\\
e(\mathcal{M}_{\xi_\lambda}) &= q^{6g-4} + \ldots + 1
\end{aligned}$$

where we have written the monomials of maximum and minimum degrees in each case. The degree of the polynomial gives the dimension of the character variety, and the coefficient (which is always 1) gives the number of irreducible components. □

Corollary 5.9.3. *Let X be a complex curve of genus $g \geq 1$. Then $e(\mathcal{M}_{-\mathrm{Id}})$, $e(\mathcal{M}_{\xi_\lambda})$, and its invariant and non-invariant part given in Cororally 5.8.7, are palindromic polynomials.*

Proof. Let $d = 6g - 4$. If we write $R(\mathcal{M}_{\xi_\lambda}) = AT + BN$, with

$$\begin{aligned} A &= (q^3 - q)^{2g-2}(q^2 + q) + (q+1)(q^2 - 1)^{2g-2} - q(q^2 - q)^{2g-2}, \\ B &= (2^{2g} - 1)q(q^2 - q)^{2g-2}, \end{aligned}$$

given in Corollary 5.8.7, then one only has to check that $q^d A(q^{-1}) = A(q)$ and $q^d B(q^{-1}) = B(q)$, which is straightforward.

The computation for $e(\mathcal{M}_{-\mathrm{Id}})$ is analogous and it is also given in [60, Section 4.4]. □

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